Advanced Calculus

Field of Real and Complex Numbers

ThinkBS: Basic Sciences in Engineering Education

Kadir Has University, Turkey

We saw that some subsets of $\mathbb{Q}$ does not have a supremum or an infimum in $\mathbb{Q}$. The property of 'having a supremum by any non-empty bounded from above' is so important that without it many results in Calculus will fail to exist.
For example, if supremum property does not hold, we can never say:

1. A continuous function on a closed interval, obtains its absolute minimum and maximum value.
2. (Intermediate Value Theorem) A continuous function defined on an interval [a, b] takes on all intermediate values.
3. (Bolzano–Weierstrass theorem) Any increasing bounded sequence (of *Real* numbers) is convergent.

So it is not only the functions, but the spaces themselves are also important.

The Field of Real Numbers

Theorem: There exists an ordered field $\mathbb{R}$ which has the least-upper-bound property. Moreover, $\mathbb{R}$ contains $\mathbb{Q}$ as a subfield.

$\mathbb{R}$ can be constructed form $\mathbb{Q}$ by means of Cauchy Sequences or by Dedekind cuts.

For further details see 'Chap. 1: Appendix'.

Archimedean Property and Densness of $\mathbb{Q}$ in $\mathbb{R}$

Archimedean Property: If $x, y \in \mathbb{R}$ and $x > 0$, then there is a positive integer n such that $nx > y$.

Densness of $\mathbb{Q}$ in $\mathbb{R}$: If $x, y \in \mathbb{R}$ and $x < y$, then there exists an $r \in \mathbb{Q}$ such that $x < r < y$.

Both of these properties, and also existence of sqaure (and also n-th) roots in $\mathbb{R}$ for example, all are results of supremum property of $\mathbb{R}$.

Field of Complex Numbers $\mathbb{C}$

As a set, define $\mathbb{C} = \mathbb{R} \times \mathbb{R}$. On $\mathbb{C}$ we define
$(a, b) + (c, d) = (a + c, b + d)$ and
$(a, b).(c, d) = (ac - bd, ad + bc)$.

Define $i = (0, 1)$. With this notation every complex number (a, b) can be written as $a + bi$ with $i^2 = -1$.

For a Complex number $z = a + bi$, we define $\Re(z) = a$ (real part) and $\Im(z) = b$ (imaginary part).

In this way, $\mathbb{C}$ is a field containing $\mathbb{R}$ which cannot be turned into an "ordered field". $\mathbb{C}$ is algebraically closed: Every non-constant polynomial with complex coefficients, has at least one root in $\mathbb{C}$.

For further details see 'Chap. 1: The Complex Field'.