Advanced Calculus

Infimum and Supremum

ThinkBS: Basic Sciences in Engineering Education

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Definitions

Suppose S is an ordered set, and $E \subseteq S$. I f there exists a $\beta \in S$ such that $x < \beta$ for every $x \in E$, we say that E is **bounded above**, and call β an **upper bound** of E. (lower bound is also defined similarly)

Consider E is bounded above and there exists an $\alpha \in S$ with the following properties :

1. α is an upper bound of E.
2. If $y < \alpha$ then y is not an upper bound of E.

In this case α is called the least upper bound or **supremum** of E, shown by $\alpha = sup(E)$

Greatest lower bound or **infimum** (inf) can be defined similarly.

sup and inf: examples

Example 1: For $A = \{\frac{1}{n} \mid n \in \mathbb{N}\} \subseteq \mathbb{Q}$, $sup(A) = 1$, $inf(A) = 0$.

Example 2: For $B = \{r \in \mathbb{Q}^{\geq 0} \mid r^2 < 2\} \subseteq \mathbb{Q}$, $inf(B) = 0$, but the supremum does not exist in $\mathbb{Q}$. (why?)

Example 3: For $C = \{r \in \mathbb{Q}^{\geq 0} \mid 2 < r^2 < 4\} \subseteq \mathbb{Q}$, $sup(B) = 2$, but the infimum does not exist in $\mathbb{Q}$. (why?)

Example 1 and 2 shows us that there are ‘some’ gaps between rational numbers and they are not a ‘continuum’.

Characteristic properties of supremum

The necessary and sufficient conditions for α to be the supremum of a nonempty set A is:

1. For every $x \in A$, $x \leq \alpha$.
2. For every number u less than α, A has an element greater than u.

The same can be formulated for the infimum.

ε formulation of the characteristic properties of supremum

The necessary and sufficient conditions for α to be the supremum of a nonempty set A is:

1. For every $x \in A$, $x \leq \alpha$.
2. For every positive number ε, A has an element x such that $\alpha - \varepsilon < x$.

The same can be formulated for the infimum.