Advanced Calculus

Introduction to Fields

ThinkBS: Basic Sciences in Engineering Education

Kadir Has University, Turkey

A field i s a set F with two operations, called addition and multiplication, which satisfy the following "field axioms":

Axioms of Addition:

1. (A1) If $x,\ y \in F$, then their sum $x + y$ is in F.
2. (A2) Addition is commutative: For all $x,\ y \in F$, $x + y = y + x$.
3. (A3) Addition is associative: For all $x,\ y,\ z \in F$, $(x + y) + z = x + (y + z)$.
4. (A4) F contains an element 0 such that $0 + x = x$ for every $x \in F$.
5. (A5) For every $x \in F$ there is an element $-x \in F$ such that $x + (-x) = 0$.

Axioms of Multiplication:

1. (M1) If $x, y \in F$, then their product xy is in F.
2. (M2) Multiplication is commutative: For all $x, y \in F$, $xy = yx$.
3. (M3) Multiplication is associative: For all $x, y, z \in F$, $(xy)z = x(yz)$.
4. (M4) F contains an element $1 \neq 0$ such that $1x = x$ for every $x \in F$.
5. (M5) For every $0 \neq x \in F$ there is an element $1/x = x^{-1} \in F$ such that $x.(1/x) = 1$.

The Distributive Law:

- (D) For all $x, y, z \in F$, $x(y + z) = xy + xz$

Ordered Fields

Let S be a set. An **order** on S is a relation, denoted by $<$, with the fol lowing two properties:

1. If x, $y \in S$ then one and only one of the statements $x < y$, $x = y$, $y < x$ is true.
2. If x, y, $z \in S$, if $x < y$ and $y < z$ then $x < z$.

The statement "$x < y$" may be read as "x is less than y".

An **ordered field** i s a field F which is also an ordered set, such that:

1. If x, y, $z \in F$, and $y < z$, then $x + y < x + z$.
2. If x, $y \in F$, and $0 < x$, $0 < y$ then $0 < xy$.

$(\mathbb{Q}, +, ., <)$, or the field or rational numbers, is an example of a ordered field. See 'Chap. 1: Fields' for more details on fields.