Abstract Algebra

Definition of Rings

ThinkBS: Basic Sciences in Engineering Education

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Sets with Two Binary Operations

In Groups we study the sets equipped with only one binary operation. Here we will add another binary operation to sets and study their properties. In its most simple case, such structures are called Rings.

As a matter of convention, from now on, the first operation on the set (with which it has become a group) will be shown by $+$ and the second one (which will make it a Ring) is denoted by . and sometimes even not written!

Definition of a Ring

A Ring $(R, +, .)$ is a set R together with two binary operations $+$ and . which we call addition and multiplication, defined on R such that the following axioms are satisfied:

- $(R, +)$ is an Abelian group.
- Multiplication is associative.
- For all a, b, $c \in R$, the left distributive law, $a.(b + c) = (a.b) + (a.c)$ and the right distributive law $(a + b).c = (a.c) + (b.c)$ hold.

A subset $S \subseteq R$ is called a subring, if with operations inherited form R, is a ring. We show this by $S \leq R$.

Rings: examples

Example 1: $(\mathbb{Z}, +, .)$, $(\mathbb{Q}, +, .)$, $(\mathbb{R}, +, .)$ and $(\mathbb{C}, +, .)$ are all rings.

Example 2: The set of integers modulo n with mod n addition and multiplication is a ring denoted by $(\mathbb{Z}_n, +, .)$.

Example 3: The set of all functions from real numbers to real numbers $\mathbb{R}^{\mathbb{R}}$ is a ring with normal addition and composition of functions.

Example 4: Let R be any ring. The set $M_n(R)$ of $n \times n$ matrices with elements from R with matrix addition and multiplication is a ring.

Basic properties of Rings

If R is a ring with additive identity 0, then for any $a, b \in R$ we have

- $0a = a0 = 0,$
- $a(-b) = (-a)b = -(ab),$
- $(-a)(-b) = ab.$

For a proof and further details look at part 4 section 18.