Abstract Algebra

Normal and Factor Groups

ThinkBS: Basic Sciences in Engineering Education

Kadir Has University, Turkey

Normal Groups

A subgroup H of a group G is **normal** if its left and right cosets coincide, that is, if $gH = Hg$ for all $g \in G$. We show this by $H \trianglelefteq G$. If a group G has no normal subgroups except than $\{1\}$ and G itself, then it is called a **simple** group.

- All subgroups of Abelian groups are normal.
- If $f : G \to G'$ is a group homomorphism, then $Ker(f)$ is a normal subgroup of G.
- Consider the sign function as a homomorphism $sgn : S_n \to \{-1, 1\}$. Using this we can see that $A_n \trianglelefteq S_n$. (Why?)
- In a group, every subgroup of index 2 is normal. (why?)

Assume $H \trianglelefteq G$ is a normal subgroup. Then the set G/H of all left (right) cosets of H becomes a group (called factor or quotient group) with group operation defined by $(aH)(bH) := (abH)$. This definition is *well-defined*.

Example: Since $3\mathbb{Z} \trianglelefteq \mathbb{Z}$, $\{3\mathbb{Z}, 1 + 3\mathbb{Z}, 2 + 3\mathbb{Z}\}$ is a group with 3 elements. Here for example
$(3\mathbb{Z}) + (1 + 3\mathbb{Z}) = (0 + 1) + 3\mathbb{Z} = 2 + 3\mathbb{Z}$ and
$(2 + 3\mathbb{Z}) + (1 + 3\mathbb{Z}) = (2 + 1) + 3\mathbb{Z} = 3 + 3\mathbb{Z} = 3\mathbb{Z}$.

It looks like that the factor group is the same group as $\mathbb{Z}_3 = \{0, 1, 2\}$ $(mod 3)$ which is not a coincidence.

Factor Groups

If we have a group homomorphism $f : G \to G'$ with $ker(f) = H$, then the map $\phi : G/H \to Im(f)$ defined by $\phi(aH) = f(a)$ is an isomorphism.

Example: Consider $f : \mathbb{Z} \to \mathbb{Z}_n$ which takes each integer to its equivalence class mod n. Here $ker(f) = n\mathbb{Z}$ and according to the above theorem $\mathbb{Z}/n\mathbb{Z} \simeq \mathbb{Z}_n$.

For further information about factor groups, look at Part 3 Section 14 of the textbook.