Abstract Algebra

Cosets and Lagrange's Theorem

ThinkBS: Basic Sciences in Engineering Education

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Cosets

Assume $H \leq G$. Let the relation $\sim_L$ be defined on G by

$$a \sim_L b \ \ iff \ \ a^{-1}b \in H$$

and let $\sim_R$be defined by

$$a \sim_L b \ \ iff \ \ ab^{-1} \in H$$

Then $\sim_L$ and $\sim_R$ are both equivalence relations on G.

Cosets

Suppose $a \in G$. The equivalence class of a consists of all $x \in G$ such that $a \sim_L x$, which means all $x \in G$ such that $a^{-1}x \in H$ which happens if and only if $a^{-1}x = h$ for some $h \in H$, or equivalently, if and only if $x = ah$ for some $h \in H$.

The sub**set** $aH = \{ah \mid h \in H\}$ of G is the left coset of H containing a, while the sub**set** $Ha = \{ha \mid h \in H\}$ is the right coset of H containing a.

Cosets: an example

We know $3\mathbb{Z} \leq \mathbb{Z}$. The left cosets of $3\mathbb{Z}$ in $\mathbb{Z}$ are:

$$0 + 3\mathbb{Z} = 3\mathbb{Z} = \{\dots, -6, -3, 0, 3, 6, \dots\}$$
$$1 + 3\mathbb{Z} = \{\dots, -5, -2, 1, 4, 7, \dots\}$$
$$2 + 3\mathbb{Z} = \{\dots, -4, -1, 2, 5, \dots\}$$

What about $3 + 3\mathbb{Z}$?
Which one of the **sets** above are subgroups?

Lagrange's Theorem

First note that every coset (left or right) of a subgroup H of a group G has the same number of elements as H. (Consider the one-to-one onto map $f : H \to aH$ from given by $f(h) = ah$).
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Definition: The number of left cosets of H in G is the index $(G : H)$ of H in G. If G is finite, then $(G : H)$ is finite and $(G : H) = |G|/|H|$, since every coset of H contains $|H|$ elements.