Abstract Algebra

Permutations, Symmetric and Alternating Group

ThinkBS: Basic Sciences in Engineering Education

Kadir Has University, Turkey

Permutation Groups

Consider the set $X = \{1, 2, 3, \dots, n\}$ with n elements. The set of all one-to-one and onto functions from X to itself with function composition is called the permutation group on n elements and is shown by S_n.

Each element of $\sigma \in S_n$ is normally shown by a two-line notation which the second line specifies the 'permuted' element:

$$\begin{pmatrix} 1 & 2 & 3 & \dots & n \\ \sigma(1) & \sigma(2) & \sigma(3) & \dots & \sigma(n) \end{pmatrix}$$

We have $S_2 = \{e = \begin{pmatrix}1 & 2\\1 & 2\end{pmatrix}, \begin{pmatrix}1 & 2\\2 & 1\end{pmatrix}\}$

and

$$S_3 = \{e = \begin{pmatrix}1 & 2 & 3\\1 & 2 & 3\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\1 & 3 & 2\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\2 & 1 & 3\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\2 & 3 & 1\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\3 & 1 & 2\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\3 & 2 & 1\end{pmatrix}\}$$

Specify the inverse of each element for each of the groups above.
Is S_3 Abelian?
How many elements will S_n have?

Can you find all subgroups of S_3?

Permutation Groups and Symmetries

Permutation groups are used in the study of symmetries of geometric shapes. For instance the group of symmetries of a square D_4 or octic group is a subgroup of S_4.

Permutation Groups and Symmetries

Permutation groups are used in the study of symmetries of geometric shapes. For instance the group of symmetries of a square D_4 or octic group is a subgroup of S_4.

They are also very useful in the study of possibility of finding roots of polynomial equations using radicals.

For further details on permutation groups, look at Part 2 Section 8 of the textbook.

Permutation Groups and Cayley's Theorem

Cayley's Theorem

Every group is isomorphic to a group of permutations.

This theorem states that symmetric groups are so rich that a copy of every group (finite or infinite - infinite symmetric groups are also defined similarly) can be found inside them.

For a proof of this, look at Part 2 Section 8 of the textbook.

Orbits and Cycles

Consider A as a non empty set and σ in the group of its permutations. We define an equivalence relation $\sim$ on A as following:

$$\textit{For } a, b \in A, \; a \sim b \textit{ iff there is an } n \in \mathbb{Z} \textit{ such that } \sigma^n(a) = b$$

The equivalence classes defined by $\sim$ are called orbits of σ.

A permutation $\sigma \in S_n$ is called a cycle if it has at most one orbit containing more than one element. The length of a cycle is the number of elements in its largest orbit.

We show cycles by single-line notation (omitting fixed points):

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 5 & 1 & 4 \end{pmatrix} = (1, 3, 5, 4)$$

Permutations and Cycles

Using cyclic notation, it can be shown that any permutation $\sigma \in S_n$ can be written as a product of disjoint cycles, for instance:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 8 & 6 & 7 & 4 & 1 & 5 & 2 \end{pmatrix} = (1,3,6)(2,8)(4,7,5).$$

By disjoint, we mean that any integer is moved by at most one of these cycles, hence no one number appears in the notations of two different cycles.

A transposition is defined as a cycle of length 2.

Theorem: No permutation in S_n can be expressed both as a product of an even number of transpositions and as a product of an odd number of transpositions.

Can you write $(2, 5, 3)$ as a product of transpositions?

Even and Odd Permutations

A transposition is defined as a cycle of length 2.

Theorem: No permutation in S_n can be expressed both as a product of an even number of transpositions and as a product of an odd number of transpositions.

Can you write $(2, 5, 3)$ as a product of transpositions?

A permutation of a finite set is **even** or **odd** according to whether it can be expressed as a product of an even number of transpositions or the product of an odd number of transpositions, respectively. We define the **sign** of an even permutation equal to 1 and of an odd one equal to -1.

The Alternating Groups

Theorem and Definition: For $n \geq 2$, the collection of all even permutations of $\{1, 2, 3, \ldots, n\}$ forms a subgroup of order $n!/2$ of the symmetric group S_n. This subgroup is called the alternating group A_n.

For more information on Alternating group, look at Part 2 Section 9.