Abstract Algebra

Homomorphisms and Isomorphisms

ThinkBS: Basic Sciences in Engineering Education

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Group Homomorphism

A function f from $(G_1, *_1)$ to $(G_2, *_2)$ is called a group **homomorphism** iff for every element a, $b \in G_1$, $f(a *_1 b) = f(a) *_2 f(b)$.

$f : (\mathbb{Z}, +) \to (\mathbb{Z}_n, +_n)$ is a homomorphism. (why?)

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A group homomorphism is called an **isomorphism**, iff it is both one-to-one and onto.

$f : (\mathbb{Z}, +) \rightarrow (2\mathbb{Z} = \{2n | n \in \mathbb{Z}\}, +)$ is an isomorphism. (why?)

For further examples look at Part 1 Section 3.

Images and Kernels

Consider a group homomorphism $f : G \to H$. We define

$$Im(f) = \{f(g) \mid g \in G\},$$
$$Ker(f) = \{g \in G \mid f(g) = e\}$$

$Im(f)$ and $ker(f)$ are called image and kernel of f respectively. One can show the following properties:

- $Im(f) \leq H$
- $ker(f) \leq G$
- f is onto iff $Im(f) = H$
- f is one-to-one iff $ker(f) = \{e\}$

Isomorphic Group

Two groups $(G_1, *_1)$ and $(G_2, *_2)$ are called isomorphic if there exist at least one isomorphism between them.

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Two isomorphic groups will have the exact same 'algebraic structure', only the label of elements will be different. If finite, their tables will also have the exact same structure, but only with different element labels.