Abstract Algebra

Solvablity by Radicals

ThinkBS: Basic Sciences in Engineering Education

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A polynomial $p(x)$ with coefficients in K is called solvable by radicals if there exists a sequence of radical extensions

$$K \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n$$

such that all the roots of $p(x)$ are in K_n.

A group G is called solvable if there exists a sequence of subgroups

$$\{e\} = G_0 \subseteq G_1 \subseteq G_2 \subseteq \cdots \subseteq G_m = G$$

such that G_i is normal in G_{i+1} and $|G_{i+1}|/|G_i|$ is prime.

Theorem: $p(x)$ is solvable by radicals iff $Gal(K_n/K)$ is solvable.

Theorem: For $n \geq 5$, the symmetric group on n letters S_n is not a solvable group.

Abel-Ruffini Theorem

Lemma: If $f(x)$ is an irreducible polynomial over $\mathbb{Q}$ of prime degree p, and if $f(x)$ has exactly $p-2$ real roots, then its Galois group is S_p.

Example: The polynomial $f(x) = 2x^5 - 5x^4 + 5$ is irreducible over $\mathbb{Q}$ (by Eisenstein's criterion: see Theorem 28.16 of textbook) which has exactly 3 real roots: one between |1 and 0, one between 0 and 2, and the third between 2 and 3. The other two roots are complex.

Lemma: If $n \geq 5$ and $Gal(L/K) = S_n$, then $Gal(L/K)$ is not solvable. The Galois group of the polynomial $f(x) = 2x^5 - 5x^4 + 5$ is isomorphic with S_5, which is not a solvable group.

See Section 49 of the textbook for more details.

Abel-Ruffini Theorem

What we have seen above means that there is at least one polynomial of degree 5 which cannot be solved using radicals. This results in saying that quintic polynomials (polynomials of degree 5) are not solvable by radicals in general.

We can also generalize this result as the Abel-Ruffini Theorem:

There exist polynomials of degree greater than or equal to 5 which are not solvable by radicals.

Note: General quintic equations can be solved using elliptic functions by reducing them into Bring-Jerrard form, the details of which is beyond our scope.