Abstract Algebra

Algebraic and Normal Extensions

ThinkBS: Basic Sciences in Engineering Education

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Algebraic Extensions

An extension field E of a field F is an algebraic extension of F if every element in E is algebraic over F. If an extension field E of a field F is of finite dimension n as a vector space over F, then as we have mentioned before, E is a finite extension of degree $[E : F] = n$ it over F.

If E is a finite extension field of a field F, and K is a finite extension field of E, then K is a finite extension of F, and

$$[K : E][E : F] = [K : F]$$

A finite extension field E of a field F is an algebraic extension of F.

For further information look at section 40.

Algebraic Extensions

Let E be an extension field of a field F, and let $\alpha_1, \alpha_2 \in E$, not necessarily algebraic over F. By definition, $F(\alpha_1)$ is the smallest extension field of F in E that contains α_1. Similarly, $(F(\alpha_1))(\alpha_2)$ can be characterized as the smallest extension field of F in E containing both α_1 and α_2. We could equally have started with α_2, so $(F(\alpha_1))(\alpha_2) = (F(\alpha_2))(\alpha_1)$. We denote this field by $F(\alpha_1, \alpha_2)$. Similarly, for $\alpha_i \in E$, $F(\alpha_1, \ldots, \alpha_n)$ is the smallest extension field of F in E containing all the α_i for $i = 1, \ldots, n$. We obtain the field $F(\alpha_1, \ldots, \alpha_n)$ from the field F by adjoining to F the elements α_i in E. $F(\alpha_1, \ldots, \alpha_n)$ can be characterized as the intersection of all subfields of E containing F and all the α_i.

Normal Extensions

A normal extension E of F is a field extension such that for every polynomial $p(x)$ with coefficients in F, if E contains one of its roots, then E contains all of its roots.

$\mathbb{C}$ is a normal extension of $\mathbb{R}$. This follows from the **Fundamental Theorem of Algebra** which states that: The field $\mathbb{C}$ of complex numbers is an algebraically closed field.

$\mathbb{Q}(\sqrt[3]{2}) = \{x + y\sqrt[3]{2} + z\sqrt[3]{4} \mid x, y, z \in Q\}$ is not a normal extension of $\mathbb{Q}$, since the complex roots of $x^3 - 2$ are not in $\mathbb{Q}(\sqrt[3]{2})$.

Normal Extensions

Theorem: L is a normal extension of K if for some polynomial $p(x)$ with coefficients in K, L contains all roots of $p(x)$.

Example: $\mathbb{Q}(\sqrt{6})$ contains $\sqrt{6}$ and $-\sqrt{6}$, which are all the roots of $p(x) = x^2 - 6$, which is a polynomial with coefficients in $\mathbb{Q}$.