Abstract Algebra

Field Extensions Cont.

ThinkBS: Basic Sciences in Engineering Education

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Back to Field Extensions

Let F be a field and let $f(x)$ be a nonconstant polynomial in $F[x]$ and take $p(x)$ to be an irreducible factor of $f(x)$ in $F[x]$.

Consider E to be equal to $F[x]/ < p(x) >$. E is a field (why?) no two different elements of F are in the same coset of $E = F[x]/ < p(x) >$.

From this we can deduce that we may consider F to be (isomorphic to) a subfield of E.

Now let α be the coset $x+ < p(x) >$ in E. Considering the evaluation homomorphism $\phi_\alpha : F[x] \to E$, we have $\phi_\alpha(f(x)) = 0$ (why?)

This means that α is a zero of $f(x)$ in E. This construction is known as Kronecker's Theorem.

Example: Let $F = \mathbb{R}$, and let $f(x) = x^2 + I$, which has no zeros in $\mathbb{R}$ and thus is irreducible over $\mathbb{R}$. Thus $< x^2 + 1 >$ is a maximal ideal in $R[x]$, so $E = R[x]/ < x^2 + 1 >$ is a field. Identifying $r \in \mathbb{R}$ with $r+ < x^2 + 1 > \in E$, we can view R as a subfield of E. Let

$$\alpha = x+ < x^2 + 1 >$$

We have

$$\alpha^2+1 = (x+ < x^2+1 >)^2+(1+ < x^2+1 >) = (x^2+1)+ < x^2+1 >= 0$$

Hence α is a zero of $f(x)$ in E. One can also show that $E \simeq \mathbb{C}$.

Algebraic and Transcendental Elements

An element α of an extension field E of a field F is algebraic over F if $f(\alpha) = 0$ for some nonzero $f(x) \in F[x]$. If α is not algebraic over F, then α is called transcendental over F.

Here again, like the case of irreducible polynomial **over a field** F, when talking about an algebraic element, we need to also specify the field.

For instance consider $\mathbb{C}$ as an extension of $\mathbb{Q}$. $\sqrt{2}$ and i are algebraic over $\mathbb{Q}$ but π and e (which is not very easy to prove) are transcendental over $\mathbb{Q}$ (but algebraic over $\mathbb{R}$ and $\mathbb{C}$ (why?)).

Characterization of Algebraic and Transcendental Elements

Let E be an extension field of a field F and let $\alpha \in E$. Let $\phi_\alpha : F[x] \to E$ be the evaluation homomorphism of $F[x]$ into E such that $\phi_\alpha(a) = a$ for $a \in F$ and $\phi_\alpha(x) = 0$. Then α is transcendental over F if and only if ϕ_α gives an isomorphism of $F[x]$ with a subdomain of E, that is, if and only if ϕ_α is a one-to-one map.

This is because the element α is transcendental over F if and only if $f(\alpha) \neq 0$ for all nonzero $f(x) \in F[x]$, which is true (by definition) if and only if $\phi_\alpha(f(x)) \neq 0$ for all nonzero $f(x) \in F[x]$, which is true if and only if $\ker(\phi_\alpha) = \{0\}$, that is, if and only if ϕ_α is a one-to-one map.