Abstract Algebra

Structure of Ideals in Rings of Polynomials

ThinkBS: Basic Sciences in Engineering Education

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Structure of Ideals in F[x]

Assume that F is a field. We give the next definition for a general commutative ring R with unity, although we are only interested in the case $R = F[x]$. Note that for a conunutative ring R with unity and $a \in R$, the set $\{ra \mid r \in R\}$ is an ideal in R that contains the element a.

If R is a commutative ring with unity and $a \in R$, the ideal $< a >= \{ra \mid r \in R\}$ of all multiples of a is called the principal ideal generated by a. An ideal N of R is a principal ideal if $N =< a >$ for some $a \in R$.

Every ideal of the ring $\mathbb{Z}$ is of the form $n\mathbb{Z}$ (why?), which is generated by n, so every ideal of $\mathbb{Z}$ is a principal ideal.

One can also show that if F is a field, every ideal in $F[x]$ is principal. (An integral domain for which every ideal is principal is called a Principal Ideal Domain or PID).

Structure of Ideals in F[x]

To show that any nonconstant polynomial $f(x)$ in $F[x]$ has a zero in some field E containing F we also need the following result:

An ideal $< p(x) > \neq 0$ of $F[x]$ is maximal if and only if $p(x)$ is irreducible over F. An ideal I in a ring R is called maximal if there are no other ideals contained between I and R.

Also if M is a maximal ideal of the commutative ring R then the quotient ring R/M is a field.