Abstract Algebra

Algebraically Closed Fields and Field Extensions

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Algebraically Closed Fields

We say that the field F is algebraically closed if every non-constant polynomial in $F[x]$ has a root in F.

Equivalently The field F is algebraically closed if and only if the only irreducible polynomials in the polynomial ring $F[x]$ are those of degree one.

We can also state that the field F is algebraically closed if and only if every polynomial $f(x)$ of degree $n \geq 1$, with coefficients in F, splits into linear factors. In other words, there are elements k, $a_1, a_2, \ldots, a_n \in F$ such that $f(x) = k(x - a_1)(x - a_2) \ldots (x - a_n)$.

Field Extensions

Consider F be a field and let $f(x)$ be a nonconstant polynomial in $F[x]$. We have seen in the examples before that when we go to a bigger field then we can find roots for polynomials. For instance $f(x) = x^2 - 2$ has no roots in $\mathbb{Q}$ but it has roots in $\mathbb{R}$. We want to see that if there exists a field E containing F such that it always contains a zero α of $f(x)$.

The answer turns to be Yes! But we first need to introduce some notions and study the structure of ideals in F[x]:

A field E is called an extension field of a field F if $F \leq E$. In this setting, we can consider E as a vector space over F (how?) and denote its dimension by $[E : F]$. We say E is a finite dimensional extension of F if $[E : F]$ is finite.