Abstract Algebra

Evaluation Homomorphisms

ThinkBS: Basic Sciences in Engineering Education

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The Evaluation Homomorphisms

Let E and F be fields, with F a subfield of E, that is $F \leq E$. Let α be any element of E, and let x be an indeterminate. Then the map $\phi_\alpha : F[x] \to E$ defined by

$$\phi_\alpha(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n) = a_0 + a_1\alpha + a_2\alpha^2 + \cdots + a_n\alpha^n$$

is a a homomorphism of $F[x]$ into E. Also, $\phi_\alpha(x) = a$, and ϕ_α maps F isomorphically by the identity map; that is, $\phi_\alpha(a) = a$ for $a \in F$. The homomorphism ϕ_α is called evaluation at at α.

Examples of Evaluation Homomorphisms

Example 1: Consider $F = \mathbb{Q} \leq \mathbb{R} = E$. Then $\phi_0 : \mathbb{Q}[x] \to \mathbb{R}$ is a homomorphism that maps every polynomial to its constant value:

$$\phi_0(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n) = a_0 + a_1.0 + a_2.0 + \cdots + a_n.0 = a_0$$

Example 2: Consider $F = \mathbb{Q} \leq \mathbb{R} = E$. Then $\phi_2 : \mathbb{Q}[x] \to \mathbb{R}$ is equal to

$$\phi_2(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n) = a_0 + 2a_1 + 2^2 a_2 + \cdots + 2^n a_n$$

Note that if $f(x)$ is a polynomial then every polynomial of the form $(x-2)f(x)$ belongs to the $ker(\phi_2)$.

Zeros of Polynomials

Let F be a subfield of a field E, and let α be an element of E. Let $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \in F[x]$, and let $\phi_\alpha : F[x] \to E$ be the evaluation homomorphism.

by $f(\alpha)$ we denote evaluation of f at α which is

$$f(\alpha) = \phi_\alpha(f(x)) = a_0 + a_1\alpha + a_2\alpha^2 + \cdots + a_n\alpha^n$$

If $f(\alpha) = 0$ we call α a zero of $f(x)$.