Abstract Algebra

Polynomial Rings and Field of Rational Functions

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Rings of Polynomials

Let R be a ring. A polynomial $f(x)$ with coefficients in R is an infinite **formal** sum

$$\sum_{i=0}^{\infty} a_i x^i = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

where $a_i \in R$ and for all but a finite number of values of i, $a_i = 0$. The a_i are called coefficients of $f(x)$. If for some $i \geq 0$ it is true that $a_i \neq 0$, the largest such value of i is called the degree of $f(x)$.

If all $a_i = 0$, then the degree of $f(x)$ is undefined although is it useful sometimes to define it as $-\infty$.

The sum is formal in the sense that we use x to pinpoint the place if a_i's (this means that $\sum_{i=0}^{\infty} a_i x^i$ is just another notation for $(a_0, a_1, a_2, a_3, ...)$) and the problem of convergence is thus irrelevant.

If $a_i = 0$ for $i > n$, then we denote $f(x)$ by $a_0 + a_1x + \cdots + a_nx^n$ and call it a polynomial. Also, if R has unity $1 \neq 0$, we will write a term $1x^k$ in such a sum as x^k and also omit all the terms with coefficient 0. a polynomial having 1 as the coefficient of the highest power of x appearing is called a **monic** polynomial.

The set $R[x]$ of all polynomials in an indeterminate x with coefficients in a ring R is a ring under polynomial addition and multiplication defined below. If R is commutative, then so is $R[x]$, and if R has unity $1 \neq 0$, then 1 is also unity for $R[x]$.

If $f(x) = \sum_{i=0}^{\infty} a_ix^i$ and $g(x) = \sum_{i=0}^{\infty} b_ix^i$ then

$$f(x) + g(x) = \sum_{i=0}^{\infty}(a_i + b_i)x^i$$

and

$$f(x)g(x) = \sum_{i=0}^{\infty}(\sum_{k=0}^{i} a_kb_{i-k})x^i$$

Rings of Polynomials of Several Variable

If R is a ring and x and y are two indeterminates, then we can form the ring $(R[x])[y]$, that is, the ring of polynomials in y with coefficients that are polynomials in x. $(R[x])[y]$ is naturally isomorphic to $(R[y])[x]$ (why?) and we refer to it as $R[x, y]$.

$R[x, y]$ is called the ring of polynomials in two indeterminates x and y with coefficients in R. The ring $R[x_1, \ldots, x_n]$ of polynomials in the n indeterminates x_i with coefficients in R is similarly defined.

One can show that if D is an integral domain, then so is $D[x]$. In particular, if F is a field, then $F[x]$ is an integral domain but not a field, because x is not a unit in $F[x]$.

Field of Rational Functions

Consider the integral domain $F[x]$ over a field F. The field of quotients of $F[x]$ is denoted by $F(x)$ and its elements are in the form of $\frac{f(x)}{g(x)}$ such that $g(x) \neq 0$ (A polynomial is nonzero if at least one of the coefficients are nonzero).

We similarly define $F(x_1, \ldots, x_n)$ to be the field of quotients of $F[x_1, \ldots, x_n]$. This field $F(x_1, \ldots, x_n)$ is called the field of rational functions in n indeterminates over F. These fields play a very important role in algebraic geometry.