Abstract Algebra

Construction of Fields

ThinkBS: Basic Sciences in Engineering Education

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The Field of Quotients

Let L be a field and D a subring of L that contains the unity. The ring D is an integral domain since it has no zero divisors. We define the field of quotients F of the integral domain D as the set of all quotients of the form $\frac{a}{b}$ with a and $b \neq 0$ both in D, which forms a subfield of L.

Example 1: Let $L = \mathbb{R}$. If $D = \mathbb{Z}$ then

$$F = \{\frac{a}{b} \mid a, b \in \mathbb{Z} \textit{ and } b \neq 0\} = \mathbb{Q}$$

Example 2: If $D = \{x + y\sqrt{2} \mid x, y \in \mathbb{Z}\}$, then

$$F = \{\frac{x + y\sqrt{2}}{z + w\sqrt{2}} \mid x, y, z, w \in \mathbb{Z} \textit{ and } z+w\sqrt{2} \neq 0\} = \{r+s\sqrt{2} \mid r, s \in \mathbb{Q}\}$$

Which is a field. (why?)

The Field of Quotients: Steps to Construct

Let D be a given integral domain, and form the Cartesian product and define

$$S = \{(a, b) \mid a, b \in D, b \neq 0\} \subseteq D \times D$$

and define two elements (a, b) and (c, d) in S to be equivalent, denoted by $(a, b) \sim (c, d)$, if and only if $ad = be$.

The relation $\sim$ between elements of the set S described above is an equivalence relation. (why?)

We show the equivalence class of an element (a, b) as $[(a, b)] := \frac{a}{b}$ and define

$$[(a, b)] + [(c, d)] = [(ad + bc, bd)] \quad or \quad \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

$$[(a, b)].[(c, d)] = [(ac, bd)] \qquad or \qquad \frac{a}{b}.\frac{c}{d} = \frac{ac}{bd}$$

The Field of Quotients: Steps to Construct

The addition and multiplication defined as above on the set of all equivalence classes of S given by $\sim$ is well-defined and also satisfies all the field axioms.

Thus we can enlarge any integral domain D to (or embedded in) a field F such that every element of F can be expressed as a quotient of two elements of D. Such a field F is called a field of quotients of D.

It can aslo be shown that every field L containing an integral domain D contains a field of quotients of D and two such fields are isomorphic. Hence we refer to F as **the** field of quotients of D.

For further details, look at Part 6 Section 26 of the textbook.