Abstract Algebra

Exact and Short Exact Sequences

ThinkBS: Basic Sciences in Engineering Education

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Exact Sequences

The pair of homomorphisms $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$ is said to be exact (at Y) if $image(\alpha) = ker(\beta)$.

A sequence $\cdots \to X_{n-1} \to X_n \to X_{n+1} \to \ldots$ of homomorphisms is said to be an exact sequence if it is exact at every X_n between a pair of homomorphisms.

With this definition we can state that if A, B and C are R-modules over some ring R, then

- The sequence $0 \to A \xrightarrow{\alpha} B$ is exact (at A) if and only if α is injective.
- The sequence $B \xrightarrow{\beta} C \to 0$ is exact (at C) if and only if β is surjective.

Short Exact Sequences

The sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is exact if and only if α is injective, β is surjective, and $image(\alpha) = ker(\beta)$. In this case we say that B is an extension of C by A.

The exact sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is called a short exact sequence.

Note that any exact sequence can be written as a succession of short exact sequences since to say

$$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$$

is exact at Y is the same as saying that the sequence

$$0 \to \alpha(X) \xrightarrow{\alpha} Y \xrightarrow{\beta} Y/ker(\beta) \to 0$$

is a short exact sequence.

The Short Five Lemma

Let α, β and γ be a homomorphism of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "\alpha"'] & B \arrow[r] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

Then

- If α and γ is injective (one-to-one), then so is β.
- If α and γ is surjective (onto), then so is β.
- If α and γ are isomorphisms, then so is β. In this case we call the two sequences isomorphic.