Abstract Algebra

Quotient Modules and Isomorphism Theorems

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Quotient Modules

Every submodule N of an R-module M is "normal" in the sense that we can always form the quotient module M/N, and the natural projection $\pi : M \to (M/N)$ is an R-module homomorphism with kernel N.

The reason for this is because a module is first of all an abelian group and so every submodule is automatically a normal subgroup and any module homomorphism is, in particular, a homomorphism of abelian groups.

To state this rigorously:

Quotient Modules

Let R be a ring, let M be an R-module and let N be a submodule of M. The (additive, abelian) quotient group M/N can be made into an R-module by defining an action of elements of R by

$$r(x+N) = (rx) + N \quad \textit{for all } r \in R,\ x+N \in M/N$$

The natural projection map $\pi : M \to (M/N)$ defined by $\pi(x) = x + N$ is an R-module homomorphism with kernel N.

Sums and Intersections

Let A, B be submodules of the R-module M. Then

- $A \cap B$ is a submodule.
- The sum of A and B defined by

  $$A + B = \{a + b \mid a \in A, b \in B\}$$

  is a submodule and is the smallest submodule which contains both A and B.

Isomorphism Theorems

- (First Theorem) Let M, N be R-modules and let $f : M \to N$ be an R-module homomorphism. Then $ker(f)$ is a submodule of M and
$$M/ker(f) \simeq Im(f)$$
.
- (Second Theorem) Let A, B be submodules of the R-module M. Then
$$(A + B)/B \simeq A/(A \cap B)$$
- (Third Theorem) Let M be an R-module, and let A and B be submodules of M with $A \subseteq B$. Then
$$(M/A)/(B/A) \simeq M/B$$