Abstract Algebra

Module Homomorphisms and Module of Homomorphisms

ThinkBS: Basic Sciences in Engineering Education

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Module Homomorphisms

Let R be a ring and let M and N be R-modules. A map $f : M \to N$ is an R -module homomorphism if it respects the R-module structures of M and N, i.e.,

- $f(x+y) = f(x) + f(y)$ for all $x, y \in M$ and
- $f(rx) = rf(x)$ for all $x \in M$ and $r \in R$.

If M and N are two R-modules we define $Hom_R(M, N)$ to be the set of all R- module homomorphisms from M into N.

A one-to-one and onto homomorphism will be called an isomorphism and if there is an isomorphism between two modules M and N, we say that they are isomorph and show it by $M \simeq N$.

Kernel and Image of an R-module can be defined as similarly to the case of groups and rings.

Module Homomorphisms: Examples

Example 1: For the ring $R = \mathbb{Z}$ the action of ring elements (integers) on any $\mathbb{Z}$-module amounts to just adding and subtracting within the (additive) abelian group structure of the module so that in this case, the second condition of a homomorphism is implied by the first condition.

for instance $f(3x) = f(x + x + x) = f(x) + f(x) + f(x) = 3f(x)$.

Hence $\mathbb{Z}$-module homomorphisms are the same as abelian group homomorphisms. Note that also $\mathbb{Z}$-modules are the same as abelian groups.

Module Homomorphisms: Examples

Example 2: Let R be a ring, let $n \in \mathbb{Z}^+$ and let $M = R^n$. For each $i \in \{1, 2, \dots, n\}$ the projection map

$$\pi_i : R^n \to R \quad by \quad \pi_i(x_1, x_2, \dots, x_n) = x_i$$

is an onto R-module homomorphism with kernel equal to the submodule of n-tuples which have a zero in position i.

Example 3: If R is a field, R-module homomorphisms are called linear transformations.

Module of Homomorphisms

Let f, g be elements of $Hom_R(M, N)$. Define $f + g$ by

$$(f + g)(m) = f(m) + g(m) \ \ for \ all \ \ m \in M$$

Then$f + g \in Hom_R(M, N)$ and with this operation $Hom_R(M, N)$ is an abelian group. If R is a commutative ring then for $r \in R$ define rf by

$$(rf)(m) = rf(m) \ for \ all \ \ m \in M$$

Then $rf \in Hom_R(M, N)$ and with this action of the commutative ring R the abelian group $Hom_R(M, N)$ is an R-module.

With addition as above and multiplication defined as function composition, $Hom_R(M, M)$ is a ring with 1 called the endomorphism ring of M and denoted by $End_R(M)$.