Abstract Algebra

Modules

ThinkBS: Basic Sciences in Engineering Education

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Modules

Let R be a ring (not necessarily commutative nor with unity 1). A left R-module or a left module over R is a set M together with

- a binary operation $+$ on M under which M is an abelian group.
- an action of R on M (that is, a map $. : R \times M \to M$) denoted by rm, for all $r \in R$ and for all $m \in M$ which satisfies
  1. $(r + s)m = rm + sm$, for all $r, s \in R$, $m \in M$
  2. $(rs)m = r(sm)$, for all $r, s \in R$, $m \in M$
  3. $r(m + n) = rm + rn$, for all $r \in R$, $m, n \in M$

     If R has a unity 1 we also impose the following condition:
  4. $1m = m$, for all $m \in M$

If R is a field, then M will be called a vector space instead of a module.

Examples

Example 1: Let R be any ring. Then $M = R$ is a left R-module, where the action of a ring element on a module element is just the usual multiplication in the ring R.

Example 2: An R-submodule of M is a subgroup N of M which is closed under the action of ring elernents, i.e., $rn \in N$, for all $r \in R$, $n \in N$. Every R-module M has the two submodules M and $0 = \{0\}$.

Example 3: Let $R = \mathbb{Z}$, let A be any abelian group (finite or infinite) and write the operation of A as $+$. We can turn A into a $\mathbb{Z}$-module by defining:

$$na = \begin{cases} a + a + \cdots + a \quad (n \ times) \ if \ \ n > 0 \\ 0 \qquad\qquad\qquad\qquad\qquad if \ \ n = 0 \\ -a - a - \cdots - a \quad (-n \ times) \ if \ \ n < 0 \end{cases}$$

Examples

Example 4: An element m of the R-module M is called a torsion element if $rm = 0$ for some nonzero element $r \in R$. The set of torsion elements is denoted $Tor(R)$. If R is an integral domain then $Tor(M)$ is a submodule of M (called the torsion submodule of M).

Example 5: If N is a submodule of M, the annihilator of N in R is defined to be

$$Ann_R(N) = \{r \in R \mid rn = 0 \ \ for\ all\ \ n \in N\}$$

The annihilator of N in R is a 2-sided ideal of R.