Abstract Algebra

Strict Skew Fields, Quaternions

ThinkBS: Basic Sciences in Engineering Education

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Strict Skew Fields

Remember that a ring R with a unity is called a division ring (or skew field) if every non-zero element is unit and we called a commutative division ring a field. We have seen examples of fields such as $\mathbb{Z}_p$ ($p \in \mathbb{Z}$ is prime) or $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$.

We also called a noncommutative division ring as a strictly skew field, but we haven't yet seen any example of such rings. This is because, first:

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Wedderburn's Theorem

Every finite division ring is a field.

and second:

In Quest of Paradise: Quaternions

We know that complex numbers are just real numbers but in two dimensions: $\mathbb{C} = \mathbb{R} \times \mathbb{R}$ and it is a field addition and multiplication defined as $(a, b) + (c, d) = (a + c, b + d)$ and $(a, b).(c, d) = (ac - bd, ad + bc)$.

If we show $(0, 1)$ by i, then we have complex numbers as we know: $(a, b) = a + bi$

Mathematicians like Gauss and Hamilton, in 19th century were trying to make $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ into a field, a task proofed later to be impossible!

In Quest of Paradise: Quaternions

Assume we introduce a second object j to extend the complex numbers to the $3D$-space and consider the triplets $a + bi + cj$ (with a, b, c real), we should first of all see what ij is.

Let $ij = \alpha + \beta i + \gamma j$. Then, $i(ij) = i(\alpha + \beta i + \gamma j) = \alpha i - \beta + \gamma ij = \alpha i - \beta + \gamma(\alpha + \beta i + \gamma j) = (\alpha\gamma - \beta) + (\alpha + \beta\gamma)i + \gamma^2 j$.

On the other hand, $i(ij) = (ii)j = -j$ (assuming associativity holds). By comparing the two expressions, we get $c^2 = -1$ for a real c, which is impossible!

In Quest of Paradise: Quaternions

It seems somehow, that if we want to multiply the triplets, ij requires a separate room for itself. Let us allocate it and call it k. Then we must admit quartets in form $a + bi + cj + dk$. To multiply triplets, we need quartets!

But how to multiply them, and to which price?

Their multiplication rules were carved by Hamilton on a stone of a bridge in Dublin in 1843: $i^2 = j^2 = k^2 = ijk = -1$.
Now ij is k as planned: $(ijk)k = -k$, so that $ij = k$ since $k^2 = -1$.
But what is ji? If we multiply $ijk = -1$ with ji, we get $ji(ijk) = -ji$ and hence $ji = -k$. Similarly, $jk = -kj = i$ and $ki = -ik = j$.

The price is the loss of commutativity! But we have found our paradise of noncommutative skew field!

Consider $\mathbb{H} = \mathbb{R}^4$ and rename the following elements as following:

$1 = (1, 0, 0, 0) \quad i = (0, 1, 0, 0)$

$j = (1, 0, 0, 0) \quad k = (0, 1, 0, 0)$

With this notation $(a, b, c, d) = a + bi + cj + dk$.

Also define the followings:

$$1a = a1 = a \ \ for\ all\ a \in \mathbb{H}$$

$$i^2 = j^2 = k^2 = -1$$

$$ij = k,\ jk = i,\ ki = j,\ ji = -k,\ kj = -i,\ ik = -j$$

Quaternions

We also define addition by

$$(a_1 + a_2 i + a_3 j + a_4 k) + (b_1 + b_2 i + b_3 j + b_4 k)$$
$$= (a_1 + b_1) + (a_2 + b_2)i + (a_3 + b_3)j + (a_4 + b_4)k$$

and multiplication by

$$\begin{aligned}
&(a_1 + a_2 i + a_3 j + a_4 k).(b_1 + b_2 i + b_3 j + b_4 k) \\
&= (a_1 b_1 - a_2 b_2 - a_3 b_3 - a_4 b_4) \\
&+ (a_1 b_2 + a_2 b_1 + a_3 b_4 - a_4 b_3)i \\
&+ (a_1 b_3 - a_2 b_4 + a_3 b_1 + a_4 b_2)j \\
&+ (a_1 b_4 + a_2 b_3 - a_3 b_2 + a_4 b_1)k.
\end{aligned}$$

With this definitions, $\mathbb{H}$ is a noncommutative skew field! (why?)