Abstract Algebra

Ring Homomorphisms, Factor Rings, Ideals

ThinkBS: Basic Sciences in Engineering Education

Kadir Has University, Turkey

A function $f : (R, +, .) \to (R', +', .')$ is called a ring homomorphism if

$$f(a+b) = f(a) +' f(b)$$
$$f(a.b) = f(a).'f(b)$$

One can show that $ker(f) = \{a \in R \mid f(a) = 0\} = f^{-1}(0)$ is a subring of R and $Im(f) = \{f(a) \mid a \in R\}$ is a subring of R'.

If f is one-to-one and onto then f is called an isomorphism. If an isomorphism exists between two rings R and R' we say that these two rings are isomorphic and show it by $R \simeq R'$.

Assume that $f : (R, +, .) \to (R', +', .')$ is a ring homomorphism with $ker(f) = H$. Then the set of additive cosets of H, shown by R/H becomes a ring, called the factor ring or the quotient ring, by binary operations defined as below:

$$(a + H) + (b + H) = (a + b) + H$$
$$(a + H).(b + H) = (a.b) + H$$

In this case, the function $\phi R/H \to Im(f)$ defined by $\phi(a + H) = f(a)$ is an isomorphism; hence $R/ker(f) \simeq Im(f)$.

Ideals

An addative subgroup I of a ring R is called an ideal if

$$aI \subseteq I, \quad Ib \subseteq I, \quad for \ all \ a, b \in R$$

. This is shown by $I \trianglelefteq R$.

Example 1: $n\mathbb{Z} \trianglelefteq \mathbb{Z}$ (why?)

Example 2: If $f : (R, +, .) \rightarrow (R', +', .')$ is a ring homomorphism, then $ker(f) \trianglelefteq R$.

Example 3: A field F has only two ideal: the trivial ideal $\{0\}$ and F itself. (why?)