Abstract Algebra

Divisors of Zero and Integral Domains

ThinkBS: Basic Sciences in Engineering Education

Kadir Has University, Turkey

Zero Divisors

If a and b are two nonzero elements of a ring R such that $ab = 0$, then a and b are **divisors of** 0 (or 0 divisors).

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$\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1/2 \\ 0 & 1/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. In the same ring, $\begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix}$ is a divisor of zero (why?)

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Example 3: $a \in \mathbb{Z}_n$ is a unit iff $gcd(a, n) = 1$ otherwise it is a divisor of zero (why?)

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Example 3: In an integral domain (or more generally in a ring with no zero divisors) the cancellation law hold:

$$(ab = ac \text{ and } a \neq 0) \text{ iff } b = c$$

Integral Domains

Example 4: An integral domain for which all of its elements (except 0 of course!) are unit, is a field.

Example 5: Every finite integral domain is a field. (why?)

For more detail, look at part 3, Sections 18 and 19 of the textbook.