Abstract Algebra

Commutative Rings, Division Rings and Fields

ThinkBS: Basic Sciences in Engineering Education

Kadir Has University, Turkey

Rings with Unity and Commutative Rings

In a ring $(R, +, .)$ if the multiplication operation has an identity element, usually shown by 1, then $(R, +, .)$ is called a ring with unity. If also . is commutative, $(R, +, .)$ is called a commutative ring.

Example: The ring $\mathbb{Z}_{10}$, has the identity element 1 and it is commutative. The ring $M_n(\mathbb{R})$ has identity matrix I_n as its identity element; it is not a commutative ring.

In $\mathbb{Z}_{10}$, the element 4 has an additive inverse 6 (why?) Is there any multiplicative inverse for 4 in $\mathbb{Z}_{10}$ (Or is there any $x \in \mathbb{Z}_{10}$ such that $x.4 = 4.x = 1$)?

Units, Division Rings and Fields

Let R be a ring with unity $1 \neq 0$. An element u in R is a **unit** of R if it has a multiplicative inverse in R. If every nonzero element of R is a unit, then R is a **division ring** (or skew field). A **field** is a commutative division ring. A noncommutative division ring is called a 'strictly skew field.'

Example: The ring of invertible $n \times n$ real matrices $GL_n(\mathbb{R})$ with matrix multiplication and addition is a division ring.