Season 1: PDE's analytical solution

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1. Introduction

A *partial differential equation* or abbreviated partial differential equation (PDE) is an equation of the form:

$$F\left(x_1, ..., x_n, u, \frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_n}, \frac{\partial^m u}{\partial x_1^{k_1} ... \partial x_1^{k_n} n}\right) = 0$$

with $F : \Omega \subseteq \mathbb{R}^p \longrightarrow \mathbb{R}$, being $p \in \mathbb{N}, n > 1, (x_1, ..., x_n) \in \Omega$, are the independant variables and $u = u(x_1, ...x_n$ is the dependant variable being $k_1 + ... + k_n = m$.

The *order* of the PDE is indicated by the highest order derivative within the equation.

Observation: Partial derivatives can be expressed as $\frac{\partial u}{\partial x_k} = u_{x_k}$. And when $n = 2$ we use the following notation: (x, y) for problems and (t, x) for space-time problems.

1.1. Classification

An PDE is linear if, u, the dependent variable and its corresponding partial derivatives appear only at first power.

We are going to study linear equations of order 2 with constant coefficients in two dimensions, which are defined by the expression:

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G(x, y)$$

where $A, B, C, D, E, F \in \mathbb{R}$ and $G(x, y)$ is a function.

We define the discriminant as $\triangle = B^2 - 4AC$ and we can classify as follow:

- Elliptical $B^2 - 4AC < 0 \Rightarrow u_{xx} + u_{yy} = F(x, y)$

- Hyperbolic $B^2 - 4AC > 0 \Rightarrow u_{tt} - c^2 u_{xx} = F(x,t)$
- Parabolic $B^2 - 4AC = 0 \Rightarrow u_t - k u_{xx} = F(x,t)$

EDPs are used, for example, to model processes that in addition to having a temporal variation, have a spatial variation such as the variation of heat over time in a solid, the distribution of populations in a certain habitat over time or the propagation of the sound of the strings of a guitar. In general the PDEs are quite difficult to solve analytically, in fact, there are no theorems of existence and uniqueness as "simple.as those studied in the initial value problems associated with the ODE, we will try to solve the PDEs corresponding to the classical problems.

2. PDEs of second order

In two dimensions and making some assumption about their solutions it is possible to solve some classes of PDE. This procedure is the so-called variable separation method.

2.1. Method: Separate variables

In this method the function sought $u(x,y)$ is assumed to be of the form $u(x,y) = F(x)G(y)$.

This assumption verify:

$$u_x(x,y) = F'(x)G(y)$$

$$u_y(x,y) = F(x)G'(y)$$
$$u_{xy}(x,y) = F'(x)G'(y)$$
$$u_{xx}(x,y) = F''(x)G(y)$$
$$u_{yy}(x,y) = F(x)G''(y)$$

and so on.

By substituting these expressions in the PDE, it is sometimes possible to reduce an PDE to an ODE system with two equations that can be solved with the usual methods.

Example

We are going to solve the PDE $u_{xx} = 4u_y$.

We assume that the solution $u(x, y)$ will be $u(x, y) = F(x)G(y)$. So:

$$u_x = F'(x)G(y) \Rightarrow u_{xx} = F''(x)G(y)$$

$$u_y = F(x)G'(y) \Rightarrow u_{yy} = F(x)G''(y)$$

Substituting in the EDP we will obtain:

$$F''(x)G(y) = 4F(x)G'(y)$$

If the terms are now grouped independently into x and y we get:

$$\frac{F''(x)}{F(x)} = \frac{4G'(y)}{G(y)}$$

The first member is a function that only depends on x and the second member is a function that only depends on y, so the only way for equality to be fulfilled is for both values to be equal to a constant $\lambda \in \mathbb{R}$, which is called the separation constant:

$$\frac{F''(x)}{F(x)} = \frac{4G'(y)}{G(y)} = \lambda$$

By equalizing each fraction to the separation constant we obtain two ordi- naria differential equations.

$$\frac{F''(x)}{F(x)} = \lambda \Rightarrow F''(x) - \lambda F(x) = 0$$

$$\frac{4G'(y)}{G(y)} = \lambda \Rightarrow 4G'(y) - \lambda G(y) = 0$$

which are solved independently. According to the sign of λ we distinguish three cases.

Case $\lambda = 0$

We have the following equations:

$$F''(x) - \lambda F(x) = 0 \Rightarrow F''(x) = 0$$

$$4G'(y) - \lambda G(y) = 0 \Rightarrow G'(y) = 0$$

We can integrate both equations and obtain:

$$F(x) = c_1 x + c_3$$

$$G(y) = c_3$$

with $c_1, c_2, c_3 \in \mathbb{R}$. So, the function $u(x, y)$ is:

$$u(x, y) = F(x)G(y) = (c_1 x + c_3)c_3 = A_1 x + B_1.$$

Case $\lambda > 0$

We can assume that $\lambda = a^2 > 0$. We have the following equations:

$$F''(x) - \lambda F(x) = 0 \Rightarrow F''(x) - a^2 F(x) = 0$$

$$4G'(y) - \lambda G(y) = 0 \Rightarrow 4G'(y) - a^2 G(y) = 0$$

The first of these equations is of the second linear order and of constant coefficients whose general solution is of the form:

$$F(x) = c_1 e^{ax} + c_2 e^{-ax}$$

The second is also linear, but of the first order and its solution is:

$$G(y) = c_3 e^{\frac{a^2 y}{4}}$$

So, the function $u(x, y)$ is:

$$u(x, y) = F(x)G(y) = (c_1 e^{ax} + c_2 e^{-ax})c_3 e^{\frac{a^2 y}{4}} = A_2 e^{ax + \frac{a^2 y}{4}} + B_2 e^{-ax + \frac{a^2 y}{4}}.$$

Case $\lambda < 0$

We can assume that $\lambda = -a^2 > 0$. We have the following equations:

$$F''(x) - \lambda F(x) = 0 \Rightarrow F''(x) + a^2 F(x) = 0$$

$$4G'(y) - \lambda G(y) = 0 \Rightarrow 4G'(y) + a^2 G(y) = 0$$

As in the previous case, the first of these equations is second-order, in this case, as the characteristic polynomial has complex roots $+ai, -ai$, its general solution is of the form:

$$F(x) = c_1 cos(ax) + c_2 sen(ax)$$

The second is of first order and its solution is:

$$G(y) = c_3 e^{\frac{-a^2 y}{4}}$$

So, the function $u(x, y)$ is:

$$u(x, y) = F(x)G(y) = (c_1 cos(ax) + c_2 sen(ax))c_3 e^{\frac{-a^2 y}{4}} = A_3 e^{\frac{-a^2 y}{4}} cos(ax) + B_3 e^{\frac{-a^2 y}{4}} sen(ax).$$

In these examples it has been seen that similar to what happens with ordinary differential equations (ODE) for which the general solution implied the existence of arbitrary constants, in this case the solutions of an EDP usually involve arbitrary functions. In general, by solving an PDE we can obtain an infinite number of solutions that will depend on those arbitrary functions. To obtain a single solution to EDP problems and as with EDO, these problems must be associated with conditions or constraints that can be of two types: initial conditions and/or boundary conditions.

2.2. Heat equation

The heat equation describes the variation in temperature in a region over time. In the case of the heat equation in one dimension, it would describe the temperature in a bar of length L with time t and is expressed as:

$$\begin{cases} u_t = \alpha^2 u_{xx} & \text{if } t > 0 \text{ and } x \in (0, L) \\ u(0, x) = f(x) & \text{if } 0 < x < L, \\ u(t, 0) & \text{if } t > 0 \end{cases}$$

The conditions $u(t, 0) = u(t, L) = 0$ are the boundary conditions and indicate that the temperature at the ends of the bar is constant and equal to 0. While the condition $u(0, x) = f(x)$ is an initial condition and indicates the temperature distribution in the bar at the initial instant. The value α is the thermal diffusivity and depends on the material that forms the bar.

We are going to use the method of separation of variables to solve this equation, for this we will assume that the solution $u(t, x)$ can be put as a product of two functions in each of the independent variables $u(t, x) = F(t)G(x)$.

Substituting in the heat equation:

$$u_t = \alpha^2 u_{xx} \Rightarrow F'(t)G(x) = \alpha^2 F(t)G''(X)$$

Obviously the null function $u \equiv 0$ is a solution of the partial differential equation, however it will only be a solution of the problem if $f(x) = 0$ which is the trivial case; so we will look for alternative solutions and therefore assume that $F(t) \neq 0$ and $G(x) \neq 0$, therefore:

$$\frac{1}{\alpha^2}\frac{F'(t)}{F(t)} = \frac{G''(x)}{G(x)}.$$

One side of equality depends only on t and the other depends only on x, therefore, for equality both members must be constant:

$$\frac{1}{\alpha^2}\frac{F'(t)}{F(t)} = \frac{G''(x)}{G(x)} = -\lambda$$

with $\lambda \in \mathbb{R}$, the separation constant and where we have chosen the minus sign by convention.

From the previous equation we get two differential equations:

$$F'(t) + \lambda\alpha^2 F(t) = 0$$

$$G''(x) + \lambda G(x) = 0$$

Boundary conditions are transformed into:

$$u(t, 0) = F(t)G(0) = 0$$
$$u(t, L) = F(t)G(L) = 0$$

and being t arbitrary, it follows that $G(0) = G(L) = 0$.

We will have the contour problem:

$$\begin{cases} G''(x) + \lambda G(x) = 0 \\ G(0) = 0 \\ G(L) = 0 \end{cases}$$

whose solution will depend on the value of the separation parameter λ. We distinguish three cases.

Case $\lambda = 0$

We have the following equations:

$$G''(x) + \lambda G(x) = 0 \Rightarrow G''(x) = 0$$

We can integrate the equation twice and obtain:

$$G(x) = Ax + B$$

with $A, B \in \mathbb{R}$.
Taking into account the boundary conditions we have:

$$G(0) = 0 \Rightarrow B = 0$$

$$G(L) = 0 \Rightarrow AL + B = 0$$

with $L \neq 0$. So, the solution is $A = B = 0$, and we get the null solution, which we have said we are not interested in.

Case $\lambda > 0$

We can assume that $\lambda = a^2 > 0$. We have the following equations:

$$F''(x) - \lambda F(x) = 0 \Rightarrow F''(x) - a^2 F(x) = 0$$

$$4G'(y) - \lambda G(y) = 0 \Rightarrow 4G'(y) - a^2 G(y) = 0$$

The first of these equations is of the second linear order and of constant coefficients whose general solution is of the form:

$$F(x) = c_1 e^{ax} + c_2 e^{-ax}$$

The second is also linear, but of the first order and its solution is:

$$G(y) = c_3 e^{\frac{a^2 y}{4}}$$

So, the function $u(x, y)$ is:

$$u(x, y) = F(x)G(y) = (c_1 e^{ax} + c_2 e^{-ax}) c_3 e^{\frac{a^2 y}{4}} = A_2 e^{ax + \frac{a^2 y}{4}} + B_2 e^{-ax + \frac{a^2 y}{4}}.$$

Case $\lambda < 0$

We can assume that $\lambda = -\mu^2 > 0$. We have the following equations:

$$G''(x) + \lambda G(x) = 0 \Rightarrow G''(x) - \mu^2 G(x) = 0$$

whose general solution is:

$$G(x) = Ae^{\mu x} + Be^{-\mu x}$$

Taking into account the boundary conditions we have:

$$G(0) = 0 \Rightarrow A + B = 0$$

$$G(L) = 0 \Rightarrow Ae^{\mu L} + Be^{-\mu L} = 0$$

The above equations form a homogeneous linear system in the unknowns A and B. The determinant of the coefficient matrix is:

$$\begin{vmatrix} 1 & 1 \\ e^{\mu L} & e^{-\mu L} \end{vmatrix} = e^{-muL} - e^{\mu L} = -2senh(\mu L)$$

and since $\mu \neq 0$ is non-zero, the only solution of the system is the trivial $A = B = 0$, which gives us for the contour problem again the null solution.

Case $\lambda > 0$

We can assume that $\lambda = \mu^2 > 0$. We have the following equations:

$$G''(x) + \lambda G(x) = 0 \Rightarrow G''(x) + \mu^2 G(x) = 0$$

whose general solution is:

$$G(x) = Acos(\mu x) + Bsen(\mu x)$$

Taking into account the boundary conditions we have:

$$G(0) = 0 \Rightarrow A = 0$$

$$G(L) = 0 \Rightarrow Acos(\mu L) + Bsen(\mu L) = 0 \Rightarrow Bsen(\mu L) = 0$$

As we do not want the null solution must be $B \neq 0$ and

$$sen(\mu L) = 0 \Rightarrow \mu L = n\pi, n \in \mathbb{Z}$$

so,

$$\mu = \frac{n\pi}{L}$$

and the value of $\lambda = \mu^2$ is

$$\lambda = \frac{n^2\pi^2}{L^2}, n \in \mathbb{N}$$

Remember that λ was an arbitrary constant, then for each value of $n \in \mathbb{N}$, we will have a possible solution of the ODE,

$$\lambda_n = \frac{n^2\pi^2}{L^2} \Rightarrow G_n(x) = B_n sen(\frac{n\pi}{L}x)$$

Note that for $n = 0$ the null would be obtained again, then we will assume $x \geq 1$.

The only values that provide solutions other than the null solution are:

$$\lambda_n = \frac{n^2\pi^2}{L^2}$$

For these values and using the other differential equation, we have:

$$F'(t) + \alpha^2\lambda^2 F(t) = 0 \Rightarrow F'(t) = +\alpha^2\frac{n^2\pi^2}{L^2}F(t) = 0$$

whose solution for each $n \in \mathbb{N}$ is of the form:

$$F_n(t) = A_n e^{-\frac{\alpha^2 n^2 \pi^2}{L^2}t}, A_n \in \mathbb{R}$$

And the solution of the EDP will be, for each n, of the form:

$$u_n(t,x) = F_n(t)G_n(x) = \left(A_n e^{-\frac{\alpha^2 n^2 \pi^2}{L^2}t}\right)\left(B_n sen\left(\frac{n\pi}{L}x\right)\right) = b_n sen\left(\frac{n\pi}{L}x\right)e^{-\frac{\alpha^2 n^2 \pi^2}{L^2}t}$$

where we have put $b_n = A_n B_n \in \mathbb{R}$.

Since the equation is linear, any linear combination of solutions is also a solution, so we will consider as a general solution in the formal sense to:

$$u(t,x) = \sum_{n=1}^{\infty} b_n sen\left(\frac{n\pi}{L}x\right)e^{-\frac{\alpha^2 n^2 \pi^2}{L^2}t}$$

Finally, using the initial condition $u(0,x) = f(x)$ is obtained:

$$u(0,x) = \sum_{n=1}^{\infty} b_n sen\left(\frac{n\pi}{L}x\right) = f(x)$$

We can calculate the value of the coefficient b_n, if we observe the expression as a Fourier development, specifically that of the odd extension of $f(x)$, therefore:

$$b_n = \frac{2}{L}\int_0^L f(x)sen\left(\frac{n\pi}{L}x\right)dx.$$

The expression of $u(t,x)$ is said to be the formal solution because we cannot assure that it is a true solution, that is, that $f(x)$ can be represented by a trigonometric series.

On the other hand, although linearity guarantees that a finite linear combination of solutions is a solution, our linear combination is infinite, so we would have to verify that it is indeed a solution (deriving two times and substituting in the corresponding equation), and this is a difficult process, although in our case it is guaranteed by the presence of the exponential term in the formal solution, since if $n \to \infty$ then $e^{-\frac{\alpha^2 n^2 \pi^2}{L^2}t} \to 0$.

Qualitatively the equation describes a process of heat diffusion through the bar, the bar dissipates heat by converging to 0 and smoothing out any irregularities $f(x)$ you may have. We apply this analysis to a concrete example.

We have used the heat equation to exemplify a specific case of application.However, there are other classic examples such as the wave equation or Laplace equation that can also be solved by this method of separation of variables. They can be interesting cases to solve and thus to practice the method.