3. Boundary value problems

Modeling with ODEs

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Basic concepts

- **First-order ODEs**
  - Let $\Omega \subset \mathbb{R}^2, F: \Omega \to \mathbb{R}$ a continuous function. Then:

$$y'(t) = F\big(t, y(t)\big)$$

is a first-order ODE in the unknown function $y(t)$. A function $y: [a, b] \to \mathbb{R}$ is called ***a solution of the ODE*** if this equation is satisfied for every $t \in [a, b] \subset \mathbb{R}$

Basic concepts

- **Autonomous:** the ODE does not depend on t explicitly.
  - $y' = F(y(t)) \rightarrow Autonomous$. E.g. $y' = y$
  - $y' = F(y(t), t) \rightarrow Non\ autonomous$. E.g. $y' = y \cdot e^{-t}$

Basic concepts

- **Boundary value problems**

Let $\Omega \subset \mathbb{R}^2, F\colon \Omega \to \mathbb{R}$ a continuous function and $y_0, y_1 \in \mathbb{R}$. Then:

$$y''(t) = F(t, y(t), y'(t))$$

$$y(a) = y_0$$
$$y(b) = y_1$$

is an *boundary value problem* for the ODE equation. A function $y\colon [a, b] \to \mathbb{R}$ is called ***a solution of the ODE*** if all equations are satisfied for every
$t \in [a, b] \subset \mathbb{R}$

Basic concepts

- **Linear versus nonlinear**
  - A linear ODE is a linear polynomial in the unknown function

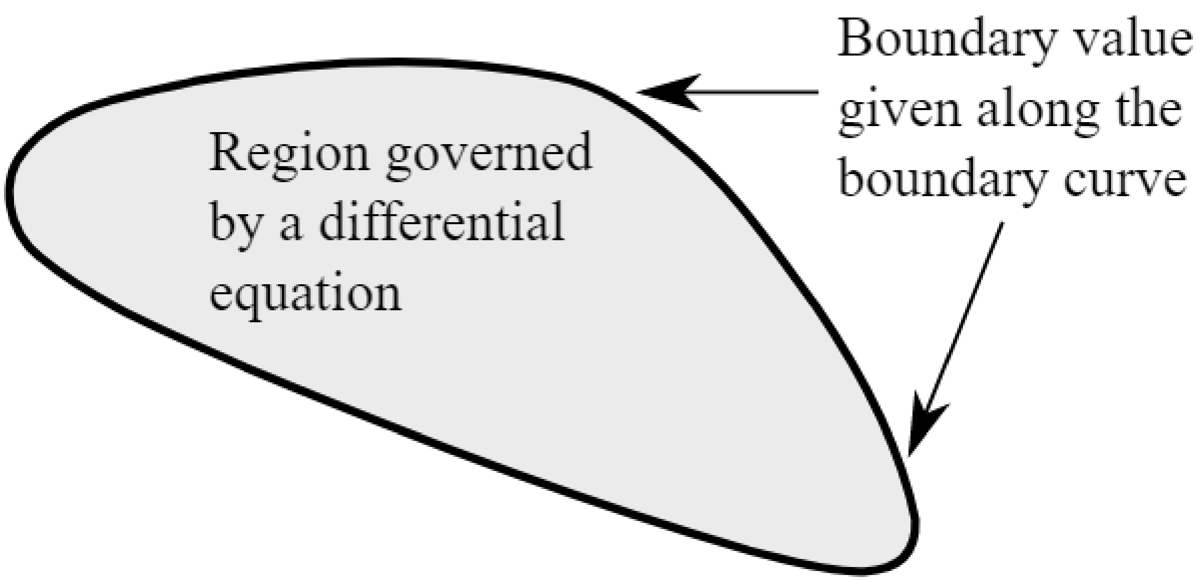
$$y^{(n)} = b(x) + a_0(x) \cdot y + a_1(x) \cdot y' + \cdots + a_{n-1}(x) \cdot y^{(n-1)}$$
$$= b(x) + \sum_{i=0}^{n-1} a_i(x) \cdot y^{(i)}$$

  - Example:
    - Body temperature
    - Clock alarm temperature

Basic concepts

- **Closed form vs. Numerical solutions**
  - Closed form: (otherwise known as analytical solution) Is a solution of the ODE in terms of "well-known" equations. Most ODEs cannot be solved in this way.
  - Numerical form: Appropriate computer algorithms are used to obtain approximations of the ODE solutions.

Boundary value problems



Boundary Value Problem. (Image source: Wikipedia)

Boundary value problems

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Definitions of derivative

- $\frac{df(t)}{dt} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}$ → Forward Euler

- $\frac{df(t)}{dt} = \lim_{\Delta t \to 0} \frac{f(t) - f(t-\Delta t)}{\Delta t}$ → Backward Euler

- $\frac{df(t)}{dt} = \lim_{\Delta t \to 0} \frac{f\left(t+\frac{\Delta t}{2}\right) - f\left(t-\frac{\Delta t}{2}\right)}{\Delta t}$ → Central differences

Boundary value problems

$$\frac{df(t)}{dt} = \lim_{\Delta t \to 0} \frac{f\left(t + \frac{\Delta t}{2}\right) - f\left(t - \frac{\Delta t}{2}\right)}{\Delta t}$$

If Δt is close to 0, we can approximate the derivative to:

$$\frac{df(t)}{dt} \approx \frac{f\left(t + \frac{\Delta t}{2}\right) - f\left(t - \frac{\Delta t}{2}\right)}{\Delta t}$$

In boundary value problems we usually need the second derivative

$$\frac{d^2 f(t)}{dt^2} \approx \frac{f'\left(t + \frac{\Delta t}{2}\right) - f'\left(t - \frac{\Delta t}{2}\right)}{\Delta t}$$

Boundary value problems

$$f''(t) = \frac{d^2 f(t)}{dt^2} \approx \frac{f'\left(t + \frac{\Delta t}{2}\right) - f'\left(t - \frac{\Delta t}{2}\right)}{\Delta t}$$

We can calculate both derivatives:

$$f'\left(t + \frac{\Delta t}{2}\right) = \frac{df\left(t + \frac{\Delta t}{2}\right)}{dt} \approx \frac{f(t + \Delta t) - f(t)}{\Delta t}$$

$$f'\left(t - \frac{\Delta t}{2}\right) = \frac{df\left(t - \frac{\Delta t}{2}\right)}{dt} \approx \frac{f(t) - f(t - \Delta t)}{\Delta t}$$

Boundary value problems

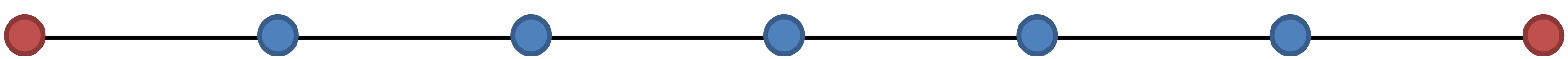
We can join both expressions:

$$f''(t) = \frac{\frac{f(t+\Delta t) - f(t)}{\Delta t} - \frac{f(t) - f(t-\Delta t)}{\Delta t}}{\Delta t} =$$

$$= \frac{f(t+\Delta t) - f(t) - f(t) + f(t-\Delta t)}{\Delta t^2}$$

$$= \frac{f(t+\Delta t) + f(t-\Delta t) - 2f(t)}{\Delta t^2}$$

Boundary value problems



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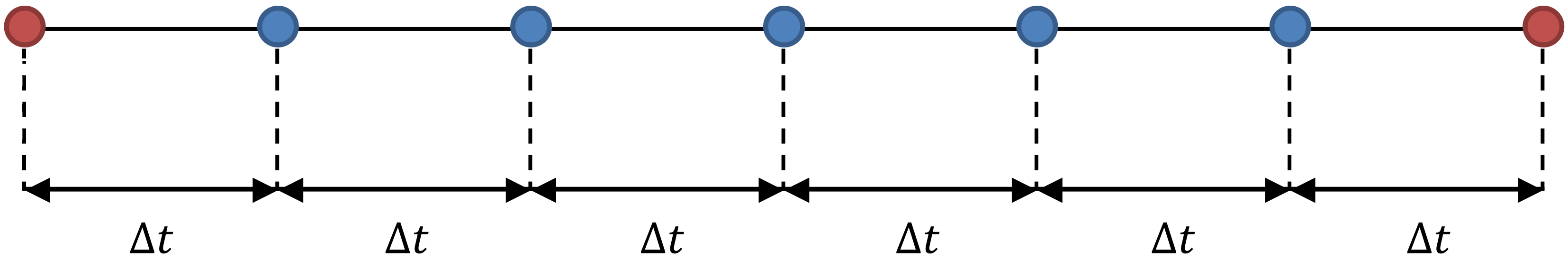

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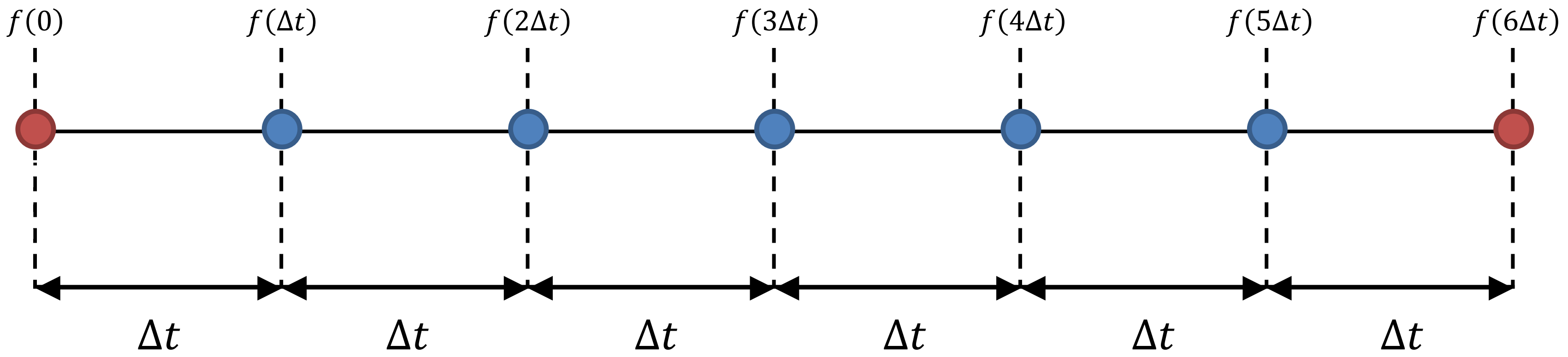

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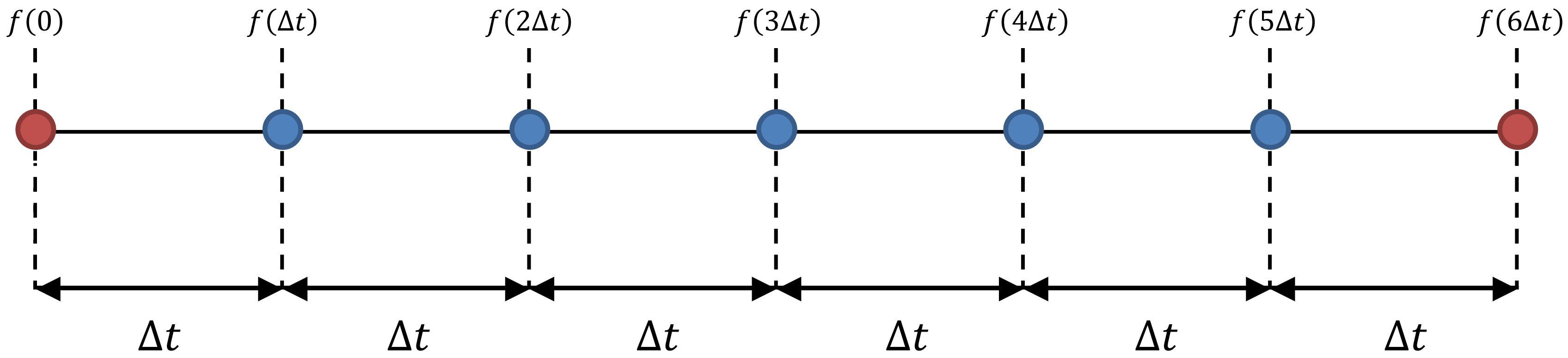

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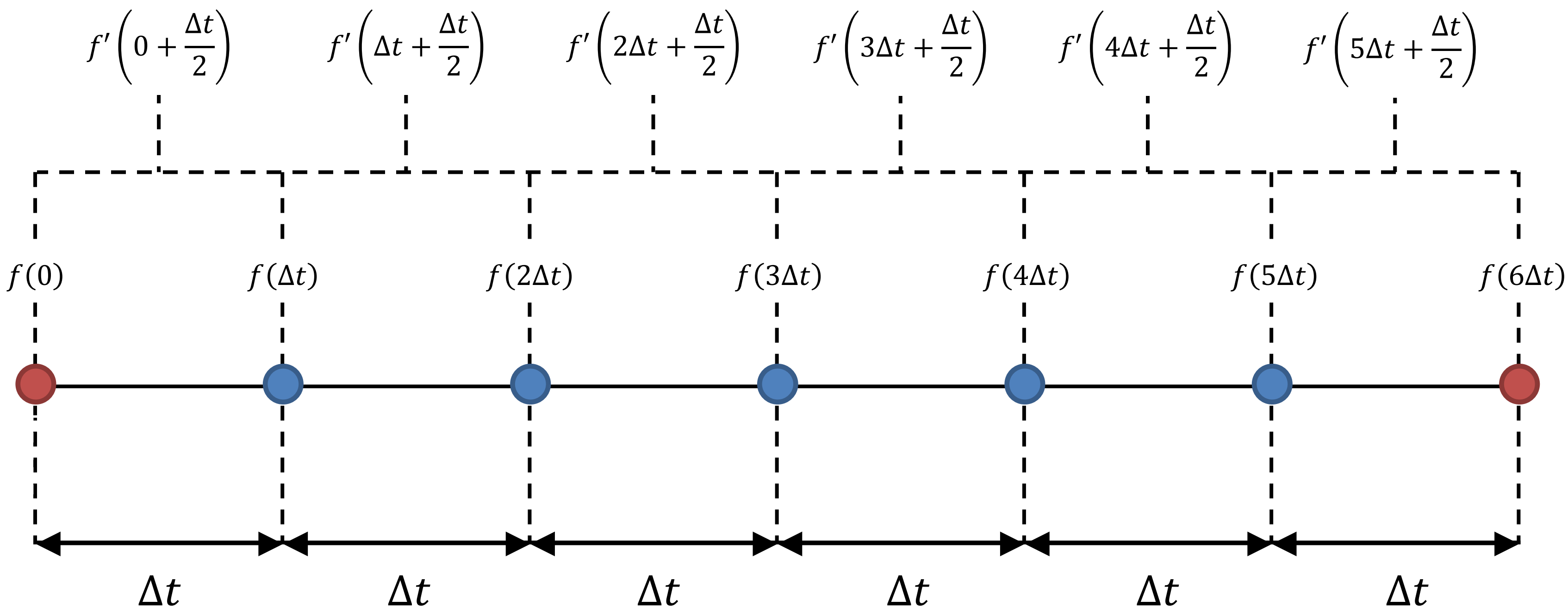

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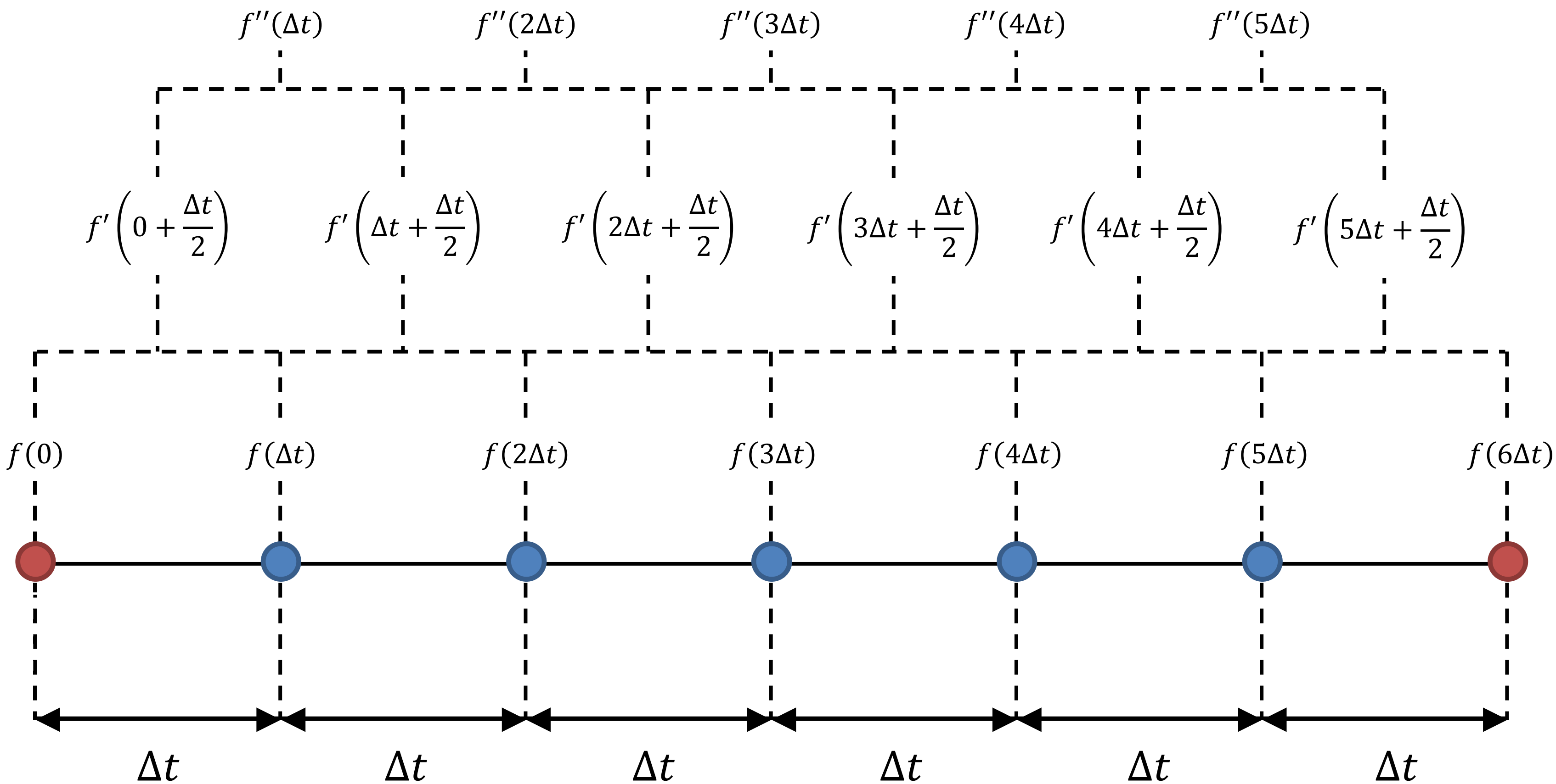

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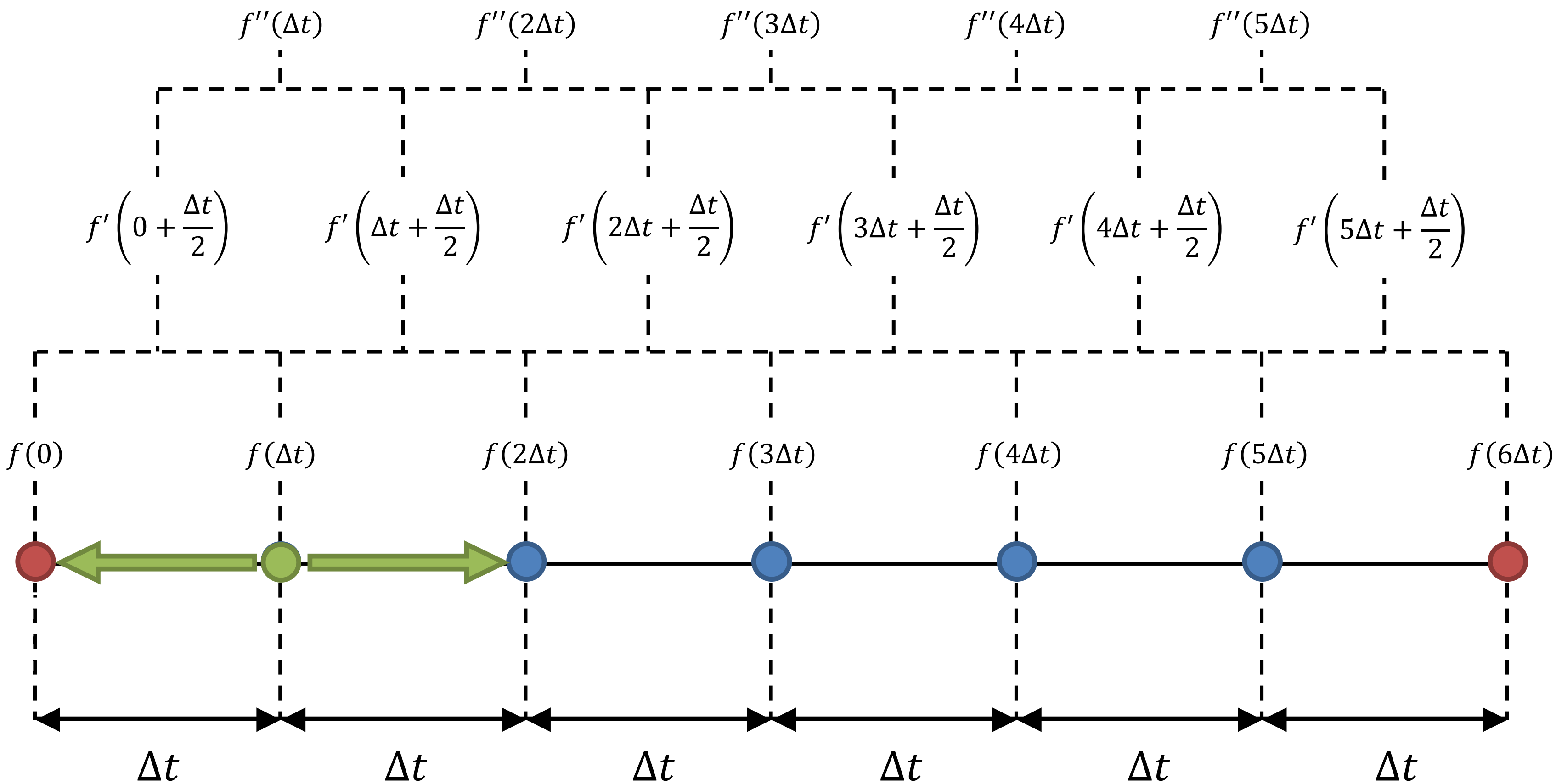

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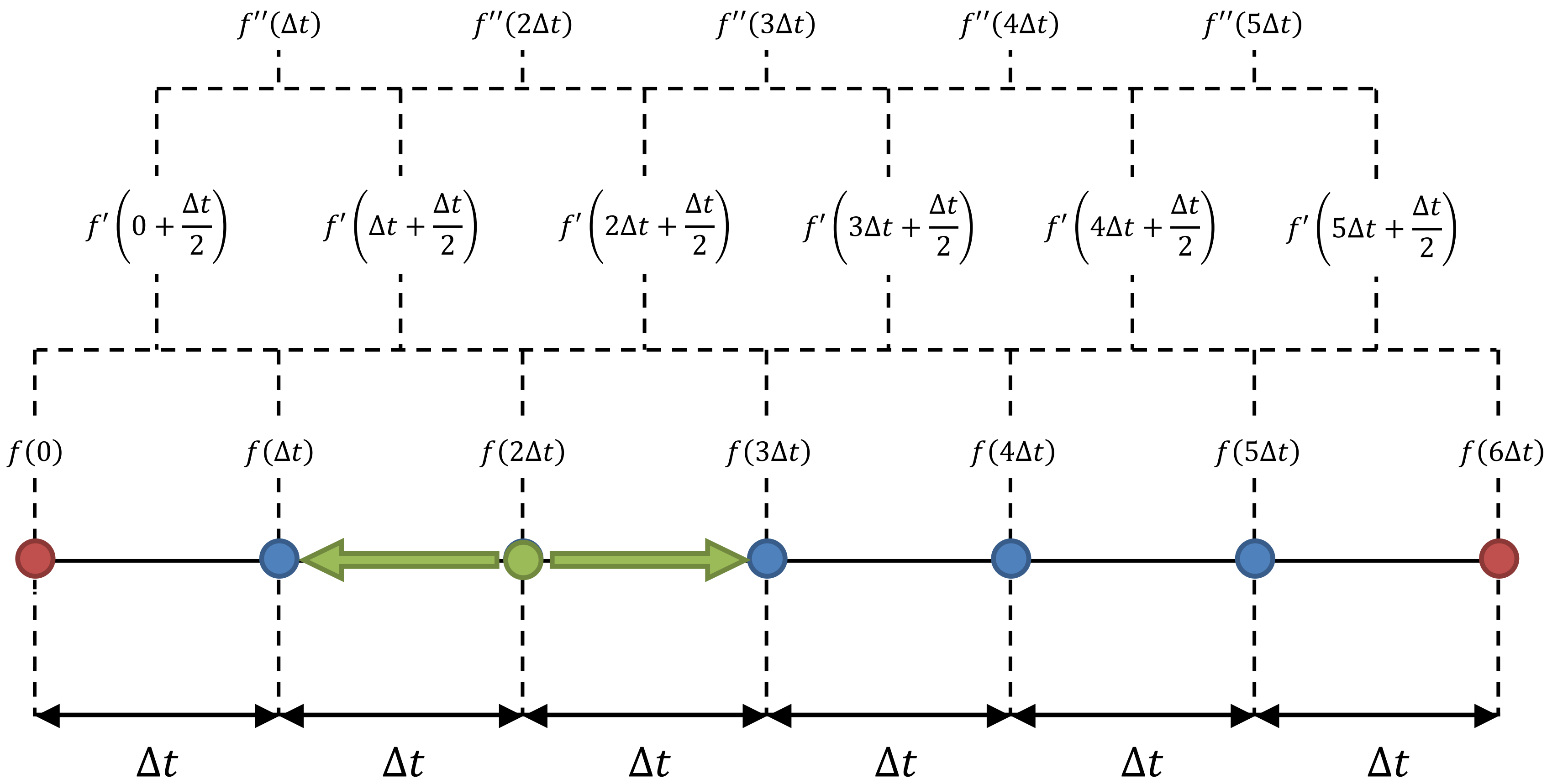

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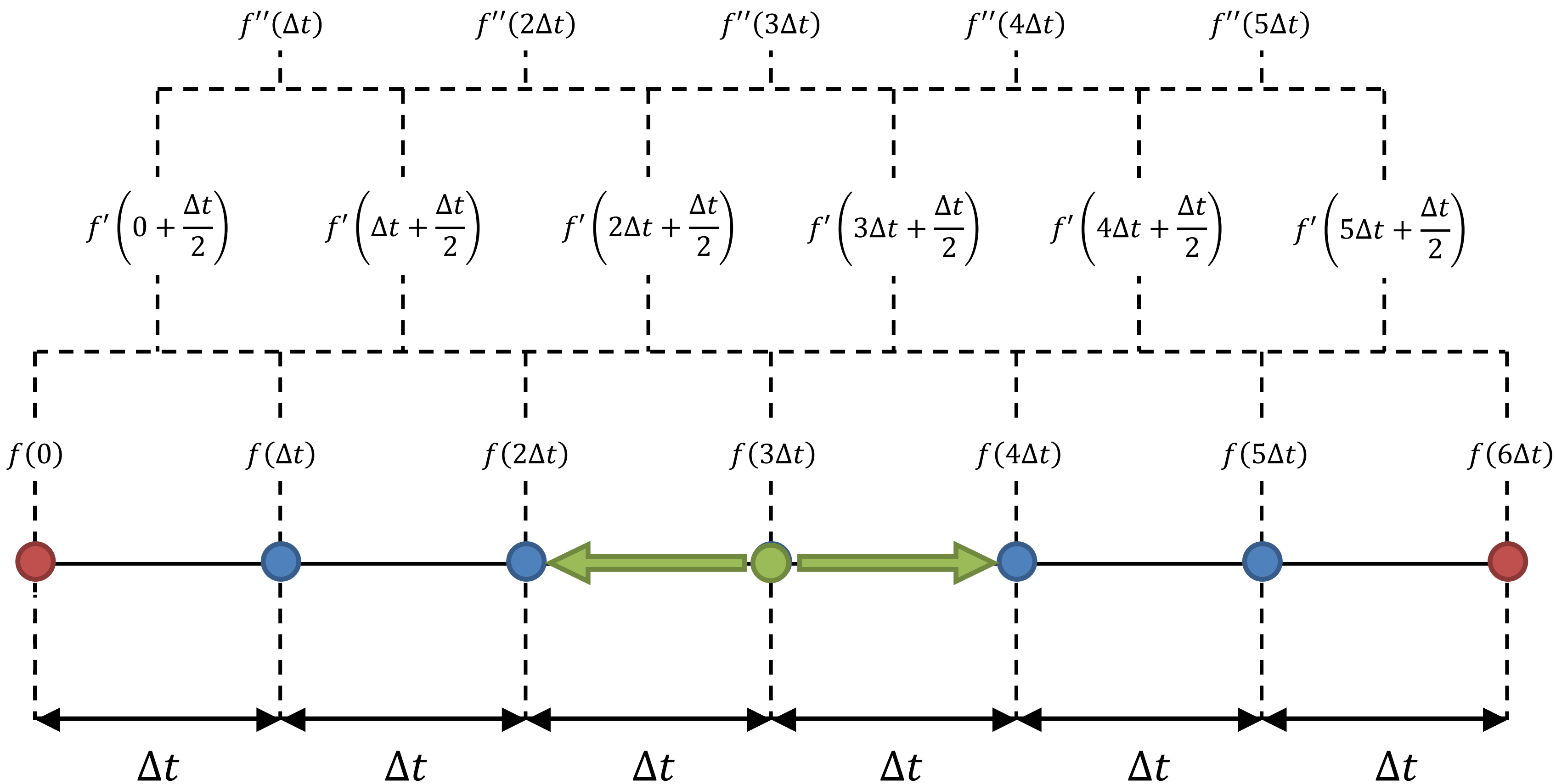

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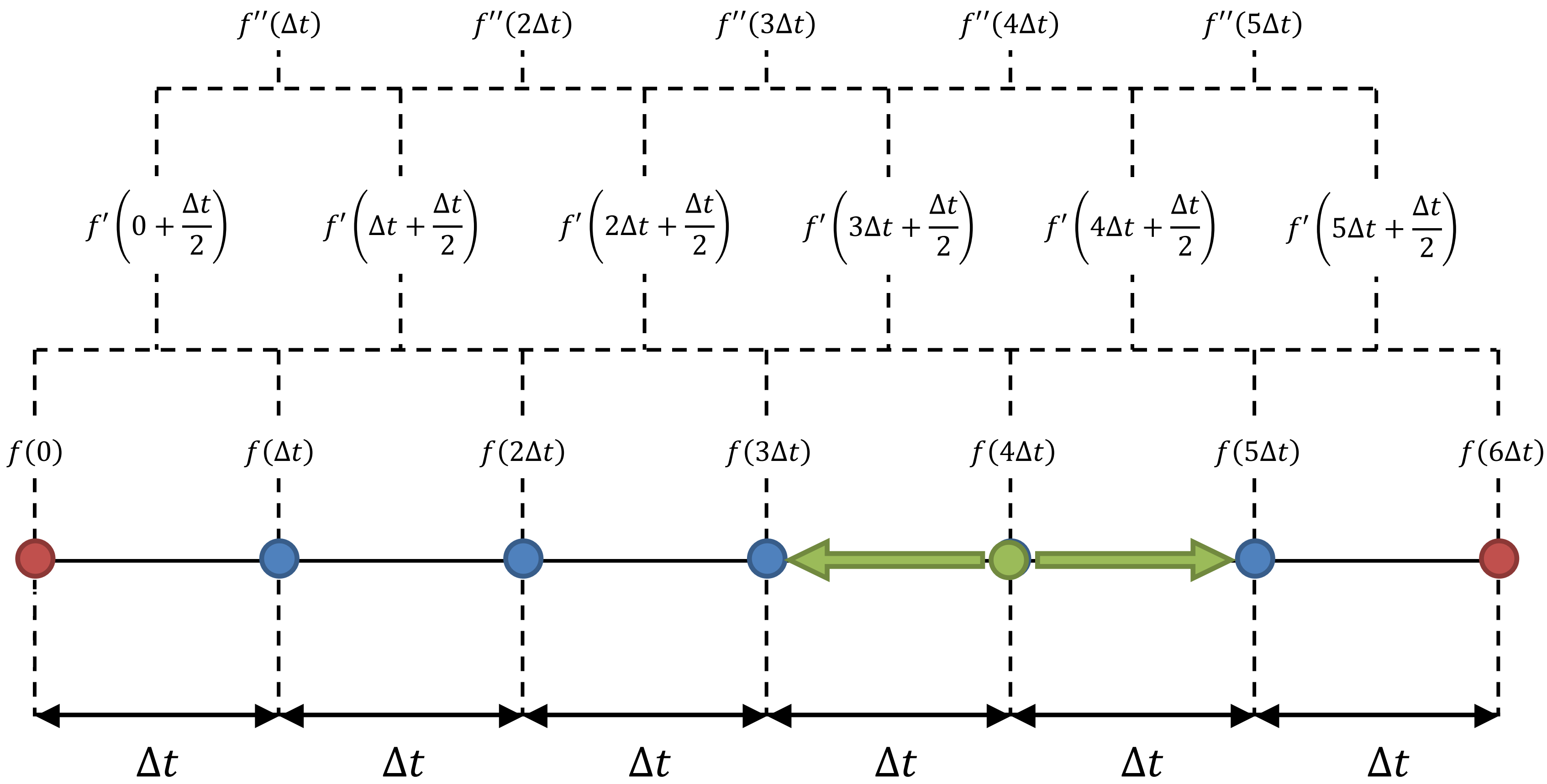

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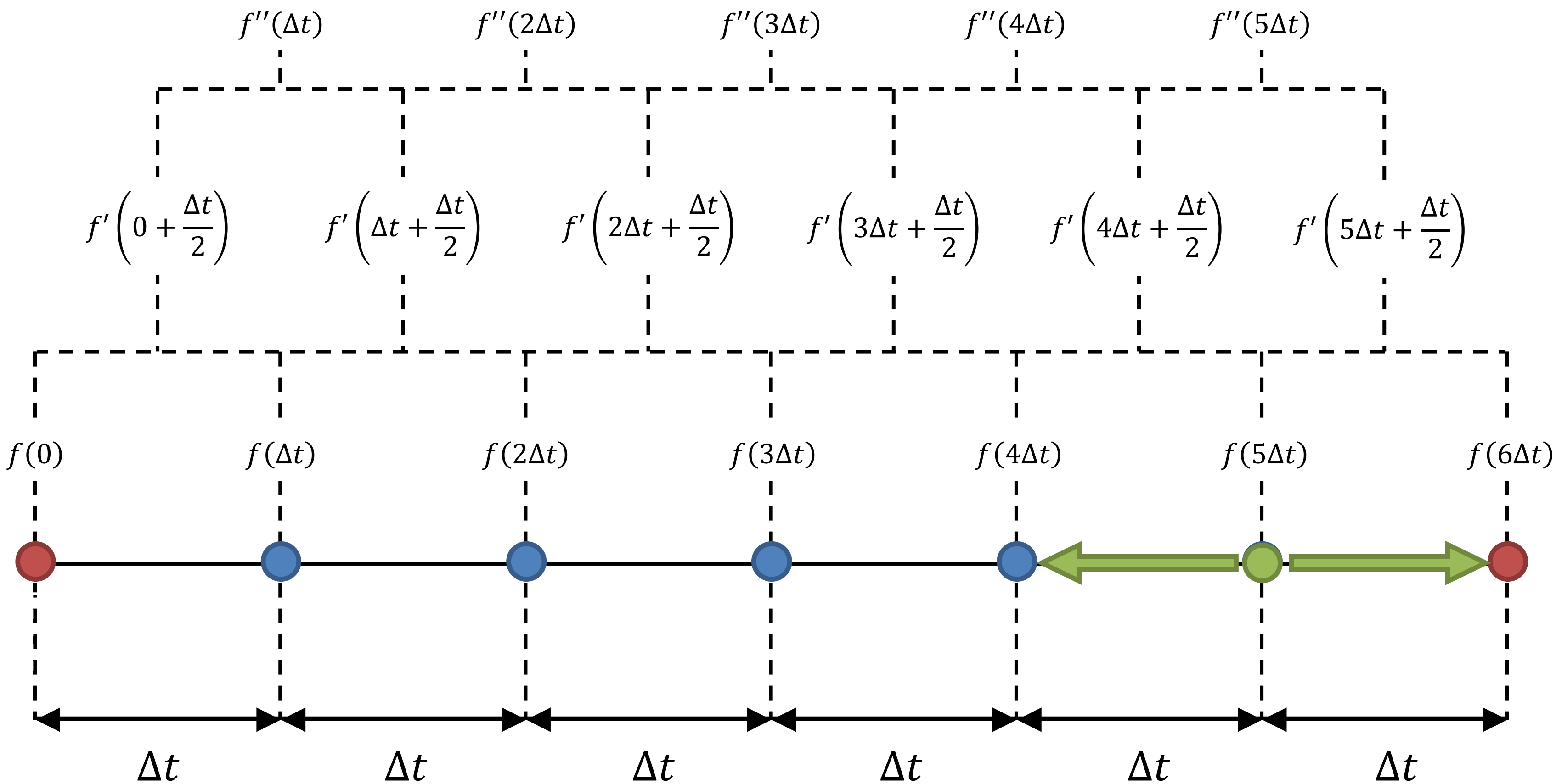
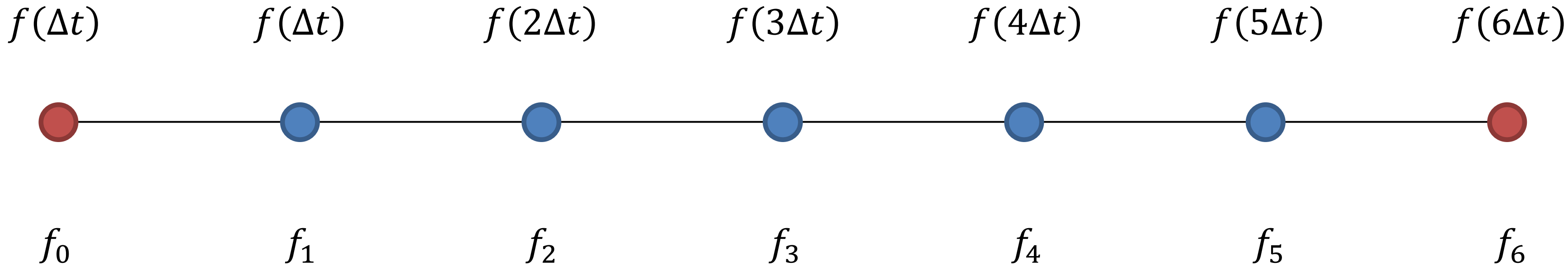


Boundary value problems

- Simplification of the notation:

$$f(n \cdot \Delta t) = f_n$$



Boundary value problems

- System of equations:
  - For each unknown (f_i) we can obtain an equation:

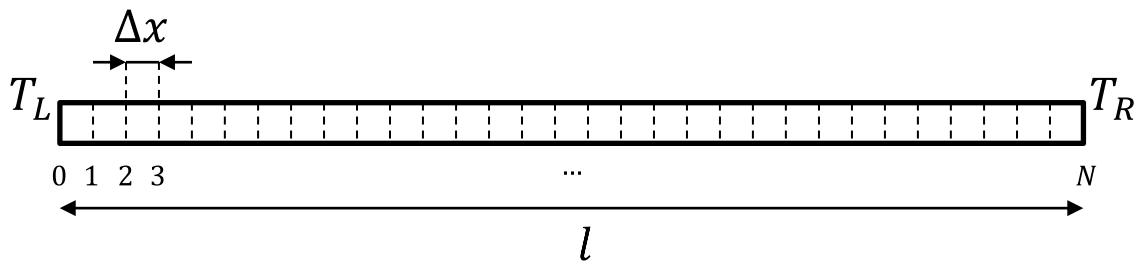
$$f_i'' = \frac{f_{i-1} + f_{i+1} - 2f_i}{\Delta t^2} \Rightarrow f_{i-1} - 2f_i + f_{i+1} = \Delta t^2 \cdot f_i''$$

  - The second derivative (f_i'') is defined in the ODE
  - We can get as many equations as unknowns

The 1D stationary Heat Equation

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}$$

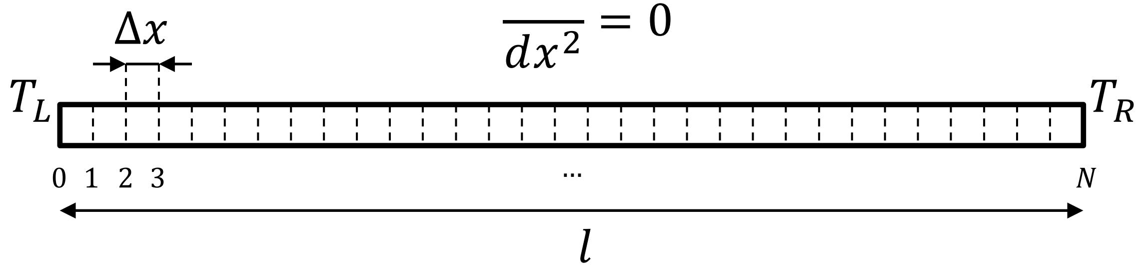
- Under steady-state conditions: $\frac{\partial T}{\partial t} = 0 \Rightarrow \frac{d^2 T}{dx^2} = 0$



- Boundary conditions
  - $T(0) = T_0 = T_L$
  - $T(l) = T_N = T_R$

The 1D stationary Heat Equation

$$\frac{d^2T}{dx^2} = 0$$

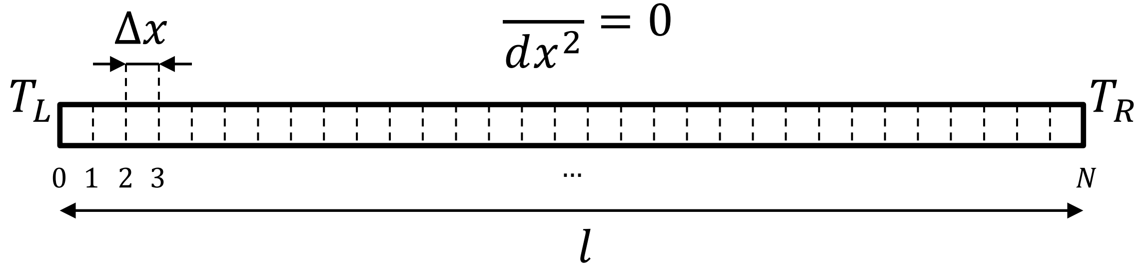


- If N=3 we have 4 equations:
  - $T(0) = T_0 = T_L$
  - $T(\Delta x) = T_1 \colon T_0 - 2T_1 + T_2 = 0$
  - $T(2\Delta x) = T_2 \colon T_1 - 2T_2 + T_3 = 0$
  - $T(l) = T_3 = T_R$
- As a product of matrices

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} T_0 \\ T_1 \\ T_2 \\ T_3 \end{pmatrix} = \begin{pmatrix} T_L \\ 0 \\ 0 \\ T_R \end{pmatrix}$$

The 1D stationary Heat Equation

$$\frac{d^2T}{dx^2} = 0$$

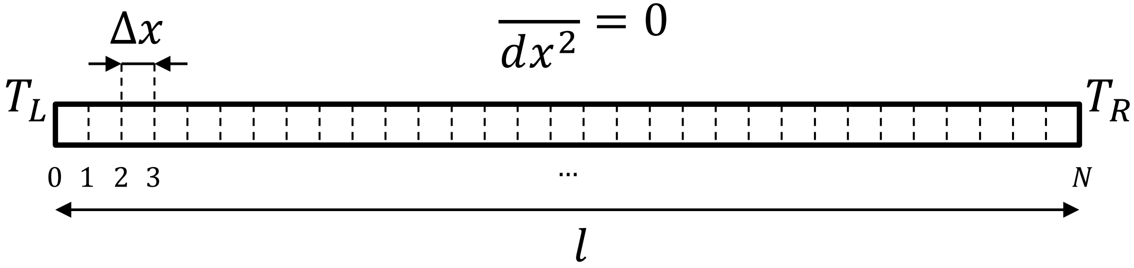


- We can extend it to the N+1 positions

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} T_0 \\ T_1 \\ T_2 \\ T_3 \\ T_4 \\ \vdots \\ T_{N-2} \\ T_{N-1} \\ T_N \end{pmatrix} = \begin{pmatrix} T_L \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ T_R \end{pmatrix}$$

The 1D stationary Heat Equation

$$\frac{d^2T}{dx^2} = 0$$



- Solving the system:

$$A \cdot T = b \Rightarrow T = A^{-1} \cdot b$$

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Example

- The gravity problem with boundary values:
  - In the instant 0, a ball is at a high of 100 meters.
  - Five seconds later, the ball touch the ground.
  - Can you calculate how high the ball was at each instant of time?

$$\frac{d^2h}{dt^2} = g$$

$$h(0) = 100$$
$$h(5) = 0$$

Resumen

- Initial value problems
  - Explicit:

$$\frac{df(t)}{dt} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t} \to f(t+\Delta t) \approx f(t) + \Delta t \cdot f'(t) \text{ (Forward Euler)}$$

  - Implicit

$$\frac{df(t)}{dt} = \lim_{\Delta t \to 0} \frac{f(t) - f(t-\Delta t)}{\Delta t} \to f(t+\Delta t) \approx f(t) + \Delta t \cdot f'(t+\Delta t) \text{ (Backward Euler)}$$

- Boundary value problems:

$$\frac{df(t)}{dt} = \lim_{\Delta t \to 0} \frac{f\left(t+\frac{\Delta t}{2}\right) - f\left(t-\frac{\Delta t}{2}\right)}{\Delta t} \to f''(t) \approx \frac{f(t-\Delta t) - 2f(t) + f(t+\Delta t)}{\Delta t^2} \text{ (Central Differences)}$$

Resumen

- For all the problems are other methods of higher order
- We have learned the most easy and typicall ones for each problem
- You can try to use the different techniques in the other problems:
  - E.g.: FE for the boundary value problem.

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