Season 1: ODE's analytical solution

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1. Introduction

1.1. First definitions

A *ordinary differential equations (ODE)* is an equation that relates a variable x, a function of that variable $y(x)$, and the derivatives of that function. In general, it can be expressed like this:

$$F(x, y, y', \dots, y^{(n)}) = 0$$

with x the *independent variable* and y the *dependent variable*

$$y''(x) + 2y'(x) + 3y(x) = 0$$

The *order* of an ODE is the highest order or degree of derivation that appears in the equation.

- $y'' + y \cdot y' = 0$, ODE of order 2.
- $y'^2 + y'' \cdot y''' = x^2$, ODE of order 3.
- $(y')^2 = y$, ODE of order 1.

If we have to solve several ODEs at the same time, they form a *system of differential equations*.

$$\begin{cases} F(x, y, y', \dots, y^{(n)}) = 0 \\ G(x, y, y', \dots, y^{(n)}) = 0 \end{cases}$$

An ordinary differential equation $F(x, y, y', \dots, y^{(n)}) = 0$ is *linear* if F is linear in the variables $y, y', \dots, y^{(n)}$. The general ODE of order n es:

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y = g(x) \tag{1}$$

An equation that is not of the above form is *nonlinear*.

- $y'' + y = e^x$ linear equation.
- $y'' + 2\sin y = 0$ nonlinear equation.

1.2. Solution of an ODE

A *solution* of an ordinary differential equation

$$F(x, y, y', \dots, y^{(n)}) = 0$$

is a function $y = z(x)$ that verify:

- $y = z(x) \in \mathcal{C}^{(n}([a, b])$,
- Verify the equation:

$$F(x, z(x), z'(x), \dots, z^{(n)}(x)) = 0$$

For example:

- $y(x) = \frac{1}{x}$ is a solution of the equation $y' = \frac{-1}{x^2}$.
- $y(x) = e^x$ is a solution of the equation $y' = 3e^x - 2y$.

In general, an ordinary differential equation can have more than one solution,even infinite solutions represented by a single expression containing constants. This expression is called *general solution*. If specific values are assigned to the contants, the solution obtained is a *particular solution*.

Finaly, it is said that we have an *Initial Value Problem (IVP)* of order one, if in addition to the ordinary differential equation, we have some complementary condition at a single point,

$$\begin{cases} y' = f(x, y) \\ y(x_0) = y_0 \end{cases}$$

If the conditions are given at different points, it is a *Boundary Problem (CVP)*.

$$(PVI) \begin{cases} y' = 3y \\ y(0) = 0 \end{cases} , \qquad (PVC) \begin{cases} y'' + y' = 0 \\ y(0) = 0, y(1) = 0 \end{cases}$$

2. ODEs of first order

2.1. Separate variables

The general form of these equations is:

$$y' = f(x)g(y)$$

To find the general solution we have to do:

$$\frac{dy}{dx} = f(x)g(y) \Leftrightarrow \frac{dy}{g(y)} = f(x)dx$$

$$\int \frac{dy}{g(y)} = \int f(x)dx + C$$

Observation: It is assumed that $g(y)$ does not cancel at any point in the interval of integration of y.

Example.

$$y'(x) = (1 + y^2)x$$
$$\int \frac{dy}{1 + y^2} = \int xdx + C$$
$$\arctan(y) = \frac{x^2}{2} + C \Leftrightarrow y = \tan\left(\frac{x^2}{2} + C\right)$$

Exercise.

$$\frac{x^2}{x-1}dx + \frac{y^2}{y+1}dy = 0$$

2.2. Homogeneous equations

The form of these homogeneous equations is $y' = f(x, y)$ where we can express $f(x, y)$ as $y' = F(\frac{y}{x})$.

To transform the given equation into an equation of separate variables, we have to do the change of variable: $y(x) = u(x)x$. Specifically, it becomes:

$$u'(x) = \frac{F(u) - u}{x}$$

Observation: $M(x, y)dx + N(x, y)dy = 0$ is homogeneous of grade n if $M(x, y)$ and $N(x, y)$ are both homogeneous of the same degree.

Similarly, if the equation can be expressed in the form $y' = G(x/y)$, change $x(y) = u(y) \cdot y$ will be made.

Example.

$$(x^3 + y^3)dx - 3xy^2 dy = 0$$

The equation is homogeneous of grade 3. If we do the change $y = ux, \quad dy = udx + xdu$, we have:

$$\begin{aligned}
(x^3 + u^3x^3)dx - 3xu^2x^2(udx + xdu) &= 0 \\
x^3(1 + u^3 - 3u^3)dx - 3x^4u^2du &= 0 \\
(1 - 2u^3)dx - 3xu^2du &= 0
\end{aligned}$$

If $x \neq 0$, and $1 - 2u^3 \neq 0$,

$$\begin{aligned}
\frac{dx}{x} &= \frac{3u^2du}{1 - 2u^3} \\
\log|x| + \frac{1}{2}\log|1 - 2u^3| &= C \Leftrightarrow \\
2\log|x| + \log|1 - 2u^3| &= C_1 \Leftrightarrow \\
x^2|1 - 2u^3| &= C_2
\end{aligned}$$

Exercises.

$$xdy - (y + \sqrt{x^2 - y^2})dx = 0$$

$$y' = -\exp(y/x) + y/x$$

$$(x - y)ydx - x^2dy = 0$$

2.3. Linear equations

A first order linear equation is one that can be expressed as follows:

$$a_1(x)y' + a_0(x)y = b(x)$$

where $a_0(x), a_1(x)$ and $b(x)$ depend only on the independent variable x, and not on y.

If it is assumed that $a_0(x), a_1(x)$ and $b(x)$ are continuous in an interval and $a_1(x) \neq 0$ in that interval, dividing by $a_1(x)$ we have:

$$y' + P(x)y = Q(x)$$

where $P(x)$ y $Q(x)$ are continous functions in the integration interval.

Thus, they are equations of the form: $y' + P(x)y = Q(x)$

The general solution is:

$$\exp\left(-\int (P(x)dx\right)\left\{C + \int [Q(x) \cdot \exp(\int P(x)dx)]dx\right\}$$

Example. The equation

$$x^2 \sin(x) - y\cos(x) = \sin(x)y'$$

is linear, because it can be written as:

$$(\sin x)y' + (\cos x)y = x^2 \sin x$$

However, the equation

$$yy' + (\sin x)y^3 = \exp(x) + 1$$

is non-linear.

Therefore, the general expression of the solution is:

$$y(x) = y_h(x) + y_p(x)$$

where $y_h(x)$ is the solution of homogeneous equation and $y_p(x)$ is the particular solution.

Homogeneous problem

$$y' + P(x)y = 0$$

$$y' + P(x)y = 0 \Rightarrow \int \frac{dy}{y} + \int P(x)dx = 0$$

$$\ln y = -\int P(x)dx + C \Rightarrow$$

$$\exp(\ln y) = \exp(-\int P(x)dx) \cdot C_1$$

$$y_h(x) = C \cdot \exp(-\int P(x)dx)$$

Particular solution We consider functions of the form:

$$y_p(x) = V(x)\exp\left(-\int P(x)dx\right) \Rightarrow$$

$$y_p'(x) = V'(x)\exp\left(-\int P(x)dx\right) - V(x)P(x)\exp\left(-\int P(x)dx\right)$$

The solution must satisfy the equation, so

$$V'(x)e^{-\int P(x)dx} - V(x)P(x)e^{-\int P(x)dx} + V(x)P(x)e^{-\int P(x)dx} = Q(x)$$

and simplifying,

$$V'(x)e^{-\int P(x)dx} = Q(x) \Rightarrow V'(x) = e^{\int P(x)dx} \cdot Q(x)$$

$$V(x) = \int Q(x)e^{\int P(x)dx}dx,$$

So,

$$y_p(x) = \exp\left(-\int P(x)dx\right)\int Q(x)\exp(\int P(x)dx))dx$$

General solution

$$y(x) = y_h + y_p = e^{-\int P(x)dx}\left\{C + \int [Q(x) \cdot e^{\int P(x)dx}]dx\right\}$$

Example.

$$y' + y = x^2$$

$P(x) = 1$ and $Q(x) = x^2$

$$y(x) = e^{-\int dx}\left[C + \int x^2 e^{\int dx}dx\right] = e^{-x}C + e^{-x}\int x^2e^xdx$$

Applying integration by parts (twice):

$$\int x^2e^xdx = \left[\begin{smallmatrix} u=x^2 \\ dv=e^xdx \\ du=2xdx \\ v=e^x \end{smallmatrix}\right] = x^2e^x - 2\int xe^xdx = \left[\begin{smallmatrix} u=x \\ dv=e^xdx \\ du=dx \\ v=e^x \end{smallmatrix}\right] = x^2e^x - 2(xe^x - \int e^xdx) = (x^2-2x+2)e^x + K$$

$$y(x) = C \cdot e^{-x} + x^2 - 2x + 2$$

Exercises.

$$\begin{cases} (x^2+1)y' + 3xy = 6x \\ y(0) = 1 \end{cases}$$

$$\begin{cases} x^2y' + xy = \sin x \\ y(1) = 2 \end{cases}$$

2.4. Exact equations

Let $P(x,y)$ and $Q(x,y)$ be two continous real functions defined in a domain $\mathcal{D}$ of the euclidean plane.

The differential equation

$$P(x,y)dx + Q(x,y)dy = 0$$

is exact if there is a real function $F(x,y)$ defined in $\mathcal{D}$, of class $C^{(1)}$ such that in said domain:

$$\frac{\partial F(x,y)}{\partial x} = P(x,y), \quad \frac{\partial F(x,y)}{\partial y} = Q(x,y).$$

The general integral is $F(x,y) = C$.

So, exists a real function $F(x,y)$ defined in $\mathcal{D}$ such that $\frac{\partial F(x,y)}{\partial x} = P(x,y), \quad \frac{\partial F(x,y)}{\partial y} = Q(x,y)$.

Integrating with respect to x in the first equality:

$$F(x,y) = \int P(x,y)dx + \phi(y) = R(x,y) + \phi(y).$$

Deriving with respecto to y,

$$\frac{\partial F(x,y)}{\partial y} = \frac{\partial R(x,y)}{\partial y} + \phi'(y) = Q(x,y).$$

This last equality allows us to determine $\phi(y)$ and write the general integral in the form $F(x,y) = C$.

Example.

$$(y\cos x + 2x\cdot e^y)dx + (\sin x + x^2e^y + 2)dy = 0$$

$$\left.\begin{array}{l} \frac{\partial P}{\partial y} = \cos x + 2xe^y \\ \frac{\partial Q}{\partial x} = \cos x + 2xe^y \end{array}\right\} \Rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

es is exact differential and therefore exists $F(x,y)$.

$$F(x,y) = \int P(x,y)dx + \phi(y) = \int (y\cos x + 2xe^y)dx + \phi(y) = y\sin x + x^2e^y + \phi(y),$$

Then, it is necessary that:

$$Q(x,y) = \frac{\partial F}{\partial y} = \sin x + x^2 e^y + \phi'(y) = \sin x + x^2 e^y + 2,$$

so that

$$\phi'(y) = 2 \Rightarrow \phi(y) = 2y + C$$

and

$$F(x,y) = y \sin x + x^2 e^y + 2y + C$$

Therefore, the solution is:

$$y \sin x + x^2 e^y + 2y = C$$

Example.

$$\frac{2x}{y^3} dx + \frac{y^2 - 3x^2}{y^4} dy = 0$$

It is exact because:

$$\frac{\partial P}{\partial y} = \frac{-6x}{y^4} = \frac{\partial Q}{\partial x}$$

$$F(x,y) = \int P(x,y)dx + \phi(y) = \int \frac{2x}{y^3} dx + \phi(y) = \frac{x^2}{y^3} + \phi(y)$$

$$Q(x,y) = \frac{\partial F}{\partial y} \Rightarrow \frac{-3x^2y^2}{y^6} + \phi'(y) = \frac{y^2 - 3x^2}{y^4},$$

$$\frac{-3x^2y^2}{y^6} + \phi'(y) = \frac{1}{y^2} - \frac{3x^2}{y^4} \Rightarrow \phi'(y) = \frac{1}{y^2},$$

so that $\phi(y) = -1/y + C$, being $F(x,y) = \frac{x^2}{y^3} - \frac{1}{y} + C$ and therefore the solution is:

$$\frac{x^2}{y^3} - \frac{1}{y} = C$$

Exercise.

$$(2xy - \sec^2(x))dx + (x^2 + 2y)dy = 0$$

Prove that $(x + 3x^3 \sin y)dx + (x^4 \cos y)dy = 0$ is not exact, but multiplying by factor x^{-1} , gives an exact differential.

$$P(x,y) = x + 3x^3 \sin y \Rightarrow \frac{\partial P}{\partial y} = 3x^3 \cos y$$

$$Q(x,y) = x^4 \cos y \Rightarrow \frac{\partial Q}{\partial x} = 4x^3 \cos y$$

so it is not exact.

Multiplying by x^{-1},

$$(1 + 3x^2 \sin y)dx + (x^3 \cos y)dy = 0,$$

$$\frac{\partial P}{\partial y} = 3x^2 \cos y = \frac{\partial Q}{\partial x}$$

and now it is exact.

$$F(x, y) = \int (1 + 3x^2 \sin y)dx + \phi(y) = x + x^3 \sin y + \phi(y)$$

$$x^3 \cos y + \phi'(y) = x^3 \cos y \Rightarrow \phi'(y) = 0 \Rightarrow \phi(y) = C,$$

obtaining

$$F(x, y) = x + x^3 \sin y + C$$

The factor x^{-1} is called *integrating factor.*

2.5. Bernoulli equations

Bernoulli equations are equations of order one that can be expressed as:

$$y' + p(x)y = q(x)y^n$$

where $p(x)$ and $q(x)$ are continous in an interval (a, b) and n is a real number.

If $n = 0$ or $n = 1$, the equation is linear and can be solved as prevously indicated. For the rest of the values of n, the variable change

$$u = y^{1-n}$$

transforms the Bernoulli equation in a linear equation.

Verification of variable change

Dividing the initial equation by y^n, we have

$$y^{-n}y' + p(x)y^{1-n} = q(x)$$

The change $u = y^{1-n} \rightarrow u' = (1 - n)y^{-n}y'$
So,

$$\frac{1}{1-n}u' + p(x)u = q(x)$$

that is linear.

Observation $y = 0$ is always solution for $n > 0$.

Example

$$y' + y = y^4$$

$n = 4$, $p(x) = 1$ and $q(x) = 1$
Suppose $y(x) \neq 0$,

$$y^{-4}y' + y^{-3} = 1 \ \rightarrow_{u=y^{-3}} \ u' - 3u = -3$$

that is linear and its solution is $u = 1 + Ce^{(-3x)}$
So, the solution is:

$$\frac{1}{y^3} = 1 + Ce^{-3x}$$

Exercises.

$$y' - 5y = -\frac{5}{2}xy^3$$

$$y' = \frac{y}{x} + y^3$$

3. Linear ODEs of order n

Finally we are going to study differential equations of order n. These equations are like this:

$$y^n(x) + a_1 y^{n-1}(x) + \cdots + a_n y(x) = f(x)$$

where the coefficients a_i are constants and $f(x)$ are real and continuous functions in $[x_0, x_1]$.

There are different types:

- If $f(x) \equiv 0$, it is an *homogeneous equation*
- If $f(x) \neq 0$, it is an *nonhomogeneous equation* (or complete).

3.1. Homogeneous equation

First, we are going to see how to solve homogeneous linear equations of order n. That is, equations of the following form:

$$y^n(x) + a_1 y^{n-1}(x) + \cdots + a_n y(x) = 0$$

We define as *characteristic polynomial* of the equation $y^n(x)+a_1y^{n-1}(x)+\cdots+a_ny(x)=0$ the polynomial $q(\lambda)=\lambda^n+a_1\lambda^{n-1}+\cdots+a_n$.

On the other hand, we define *characteristic equation to*:

$$\lambda^n+a_1\lambda^{n-1}+\cdots+a_n=0$$

The solution of the homogeneous equation is related to the solution of the characteristic equation.

If we solve the characteristic equation, we can obtain simple real solutions, real with multiplicity greater than one or complex. The solution to the equation will depend on what the roots are like.

1. If we have a real root $\lambda_0 \in \mathbb{R}$, the solution will be of the form:

$$y_1(x)=e^{\lambda_0 x}$$

2. If the polynomial has a real root $\lambda_1 \in \mathbb{R}$ of multiplicity greater than one, for example two, we will have two solutions of the form:

$$y_1(x)=e^{\lambda_1 x}, \quad y_2(x)=xe^{\lambda_1 x}$$

3. If we have two complex conjugate roots, $\lambda_1=\alpha+\beta\, i,\ \lambda_2=\alpha-\beta\, i; \alpha,\beta\in\mathbb{R}$, the solutions will be of the form:

$$y_1(x)=\cos(\beta x)e^{\alpha x}, \quad y_2(x)=\sin(\beta x)e^{\alpha x}$$

The final solution of the homogeneous equation will be a linear combination of the system of independent solutions previously obtained.

Example.

$$y''-y'-20y=0$$

1. The characteristic polynomial is $q(\lambda)=\lambda^2-\lambda-20$.

2. We calculate the solutions of $q(\lambda)=0$. $\Delta=1+80=81>0$ There are two different real solutions: $\lambda_1=-4$ and $\lambda_2=5$.

3. The independent solutions are:

$$y_1(x)=e^{-4x}, \quad y_2(x)=e^{5x}.$$

4. Therefore, we do a linear combination and the final solution is:

$$y(x) = C_1 e^{-4x} + C_2 e^{5x},$$

with C_1 and C_2 arbitrary real constants.

If we want to find the solution of a complete equation, we have to calculate a particular solution,in addition to the homogeneous one. We are going to do it with the method of indeterminate coefficients.

3.2. Method of indeterminate coefficients

To solve the following linear equation of constant coefficients

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 = f(x),$$

the functions $f(x)$ have to be as follows:

- a polynomial, $f(x) = P_n(x) = b_n x^n + \cdots + b_1 x + b_0$,
- an exponential function, $f(x) = ae^{bx}$,
- a sine function, $f(x) = a\sin(b \cdot x)$ or a cosine function, $f(x) = a\cos(b \cdot x)$
- an additive or multiplicative combination of the others.

Depending on the expression of $f(x)$, one of the solutions proposed in the following table is chosen, with indeterminate coefficients that must be adjusted later.

independent term $f(x)$	solution type $y_p(x)$	root
a	α	0
ax^n	$\alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n = P_n(x)$	0
$P_n(x)$	$\alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n$	0
ae^{bx}	αe^{bx}	b
$P_n(x)e^{bx}$	$(\alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n)e^{bx}$	b
$a\sin(bx)$	$\alpha\cos(bx) + \beta\sin(bx)$	$\pm ib$
$a\cos(bx)$	$\alpha\cos(bx) + \beta\sin(bx)$	$\pm ib$
$ce^{ax}\sin(bx)$	$(\alpha\cos(bx) + \beta\sin(bx))e^{ax}$	$a \pm ib$
$ce^{ax}\cos(bx)$	$(\alpha\cos(bx) + \beta\sin(bx))e^{ax}$	$a \pm ib$

Observation. If the independent term $f(x)$ has a root that coincides with one of the roots of the characteristic equation, with multiplicity m, the solution is that of the table multiplied by x^m.

Example.

$$y'' + 2y' + y = x^2 e^{3x}$$

1. Function $f(x)$ has a root in $\lambda = 3$, which does not coincide with those of the characteristic polynomial ($q(\lambda) = \lambda^2 + 2\lambda + 1 = (\lambda + 1)^2$).
2. This function $f(x)$ is of the type "power"by .$^{\text{ex}}$ponential", so:
$$y_p(x) = (a_0 + a_1 x + a_2 x^2)e^{3x}.$$
3. Deriving and substituting in the differential equation, we have:
$$[(16a_0 + 8a_1 + 2a_2) + (16a_1 + 16a_2)x + 16a_2 x^2]e^{3x} = x^2 e^{3x}.$$
4. We match the proposed terms with the real values of the coefficients in the given equation,
$$\left.\begin{array}{rcrcrcl} 16a_0 & + & 8a_1 & + & 2a_2 & = & 0 \\ & & 16a_1 & + & 16a_2 & = & 0 \\ & & & & 16a_2 & = & 1 \end{array}\right\}$$
5. Solving: $a_2 = 1/16, a_1 = -1/16$ y $a_0 = 3/128$.
6. The particular solution is:
$$y_p(x) = \left(\frac{3}{128} - \frac{1}{16}x + \frac{1}{16}x^2\right) e^{3x}$$

Exercise.

$$y'' - y = 3e^{-x}$$

When the independent term is sum of functions $f(x) = f_1(x) + \cdots + f_r(x)$, each particular solution is calculated separately and added by the superposition principle.

The general solution of a complete equation is formed by calculating the homogeneous solution $y_h(x)$, the particular ones and adding them. That is to say:

$$y(x) = y_h(x) + y_{p1}(x) + \cdots + y_{pr}(x),$$

.

Example.

$$y''' + 2y'' - 4y' - 8y = xe^{2x}$$

The general solution is $y = y_h + y_p$.

The associated characteristic polynomial is $q(\lambda) = \lambda^3 + 2\lambda^2 - 4\lambda - 8$, and their roots are $\lambda_1 = 2$,$\lambda_2 = -2$ double. So, the homogeneous solution is:

$$y_h(x) = C_1 e^{2x} + C_2 e^{-2x} + C_3 x e^{-2x}$$

On the other hand, the independent term is $b(x) = xe^{2x}$, $a = 2$, $b = 0$. As $a = 2$ is root of the characteristic polynomial with multiplicity $s = 1 \Rightarrow y_p(x) = xe^{2x}(\alpha + \beta x)$

Deriving successively:

$$y_p'(x) = \alpha e^{2x} + 2\alpha x e^{2x} + 2\beta x e^{2x} + 2\beta x^2 e^{2x}$$
$$y_p''(x) = 4\alpha e^{2x} + 4\alpha x e^{2x} + 2\beta e^{2x} + 8\beta x e^{2x} + 4\beta x^2 e^{2x}$$
$$y_p'''(x) = 12\alpha e^{2x} + 8\alpha x e^{2x} + 12\beta e^{2x} + 24\beta x e^{2x} + 8\beta x^2 e^{2x}$$

Substituting into the equation and grouping:

$$16\alpha e^{2x} + 40\beta x e^{2x} + 16\beta e^{2x} - 8\beta e^{2x} = xe^{2x},$$
$$(16\alpha + 16\beta)e^{2x} + 32\beta x e^{2x} = xe^{2x}.$$

Solving:

$$(16\alpha + 16\beta) = 0,\ 32\beta = 1 \Longrightarrow \beta = \frac{1}{32},\ \alpha = \frac{-1}{32}$$

$$y_p(x) = -\frac{1}{32}xe^{2x} + \frac{1}{32}x^2 e^{2x}$$

$$y = C_1 e^{2x} + C_2 e^{-2x} + C_3 x e^{-2x} - \frac{1}{32}xe^{2x} + \frac{1}{32}x^2 e^{2x}$$

3.3. Cauchy's problem

The general solution of an linear EDO depends on several arbitrary constants $C_i \in \mathbb{R}$. To determine these constants, we have to impose additional conditions. If the conditions are at the same point, we have an Initial Value Problem, also called *Cauchy's problem*.

Example.

$$\text{PVI} \equiv \begin{cases} y'' - 2y' + 5y & = 0 \\ y(0) = 2 & y'(0) = 0 \end{cases}$$

1. We calculate the characteristic polynomial, $q(\lambda) = \lambda^2 - 2\lambda + 5$
2. Its roots, $(\Delta = 4 - 20 = -16 < 0)$ are complex conjugated: $\lambda_1 = 1 + 2i$ y $\lambda_2 = 1 - 2i$
3. The general solution is

$$y(x) = e^x (C_1 \cos(2x) + C_2 \sin(2x))$$

4. And imposing the initial conditions:

$$\begin{aligned} y(0) &= 2 \Rightarrow C_1 = 2 \\ y'(0) &= 0 \Rightarrow 2C_2 + C_1 = 0 \end{aligned}$$

   So, we obtain $C_1 = 2$ y $C_2 = -1$.

   The final solution of the Cauchy's problem is:

$$y(x) = e^x \big(2\cos(2x) - \sin(2x)\big).$$