Advanced Calculus

Partition of Unity

ThinkBS: Basic Sciences in Engineering Education

Kadir Has University, Turkey

Partition of Unity

Let X be a topological space. A family $\{\tau_\lambda\}_{\lambda\in\Lambda}$ of continuous functions $\tau_\lambda : X \to [0,1]$ is called a partition of unity if

1. It is "locally finite", in the sense that every point $x \in X$ has a neighborhood in which the τ_λ vanish for all but a finite number of λ.
2. For every $x \in X$ we have

$$\sum_{\lambda\in\Lambda} \tau_\lambda(x) = 1$$

Partition of Unity

The partition of unity is said to be subordinate to a given open cover $\mathcal{U}$ of X if for every λ the support of τ_λ i.e. the closure

$$Supp\ \tau_\lambda = \overline{\{x \in X \mid \tau_\lambda(x) \neq 0\}}$$

is entirely contained in one of the sets of the covering.

Integration via Partitions of Unity

Let M be an oriented n-dimensional manifold, and let $\{\tau_i\}_{i\in\mathbb{N}}$ be a partition of unity with each *supp* τ_i contained in the chart domain U_i of an orientation-preserving chart (U_i, h_i). Then any n-form ω can be written as a locally finite sum $\omega = \sum_{i=1}^{\infty} \omega_i$, where

$$\omega_i = \tau_i.\omega$$

Let $\alpha_i : h_i(U_i) \rightarrow \mathbb{R}$ denote the downstairs component function

$$a_i = \omega_{1\ldots n} \ o \ h_i^{-1}$$

in terms of (U_i, h_i).

Integration via Partitions of Unity

In this situation, ω is integrable if and only if each a_i is integrable on $h_i(U_i)$ and

$$\sum_{i=1}^{\infty} \int_{h_i(U_i)} |a_i| dx < \infty$$

Then

$$\int_M \omega = \sum_{i=1}^{\infty} \int_{h_i(U_i)} a_i dx$$