Advanced Calculus

Integration of Forms and Stokes' Theorem

ThinkBS: Basic Sciences in Engineering Education

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Integration of Forms

Suppose ω is a k-form in an open set $E \subseteq \mathbb{R}^n$, Φ is a k-surface in E, with parameter domain $D \subseteq \mathbb{R}^k$, and Δ is the k-surface in $\mathbb{R}^k$, with parameter domain D, defined by $\Delta(\mathbf{u}) = \mathbf{u}$ $(\mathbf{u} \in D)$. Then

$$\int_{\Phi} \omega = \int_{\Delta} \omega_{\Phi}$$

To give the general result, one needs to study the chains and orientations. The reader is encouraged to study 'Chapter 10. Simplexes and Chains' for details.

Stokes' Theorem

If Ψ is a k-chain of class $\mathcal{C}''$ (two times continuously differentiable) in an open set $V \subseteq \mathbb{R}^m$ and if ω is a $(k-1)$-form of class $\mathcal{C}'$ in V, then

$$\int_{\Psi} d\omega = \int_{\partial\Psi} \omega$$

The case $k = m = 1$ is nothing but the fundamental theorem of calculus (with an additional differentiability assumption) . The case $k = m = 2$ is called Green's theorem, and $k = m = 3$ gives the so called "divergence theorem " of Gauss. The case $k = 2$, $m = 3$ is the one originally discovered by Stokes.