Advanced Calculus

Differential Forms and Differentiation of Forms

ThinkBS: Basic Sciences in Engineering Education

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Differential Forms

A differential form of degree k, or simply a k-form, on a manifold M is a correspondence ω that assigns to every $p \in M$ an alternating k-form $\omega_p \in Alt^k \ T_pM$ on the tangent space at p. The vector space of differentiable k-forms on M is denoted by $\Omega^k M$.

A differential form assigns an ω_p to each $p \in M$, and ω_p in turn assigns a number to each k-tuple of vectors in T_pM.

A differential 1-form is a function that sends paths to real numbers and which can be expressed as a path integral in the following notation:

$$\int_C fdx + gdy$$

The name of this particular differential 1-form is $fdx + gdy$.

Differential Forms: General Definition

Suppose E is an open set in $\mathbb{R}^n$. A differential form of order $k \geq 1$ in E (briefly, a k-form in E) is a function ω , symbolically represented hy the sum

$$\omega = \sum a_{i_1 \ldots i_k}(\mathbf{x}) dx_{i_1} \ldots dx_{i_k}$$

(the indices $i_1, \ldots, i_k$ range independently from 1 to n), which assigns to each k-surface Φ in E a number $\omega(\Phi) = \int_\Phi \omega$, according to the rule

$$\int_\Phi \omega = \int_D \sum a_{i_1 \ldots i_k}(\Phi(\mathbf{u})) \frac{\partial(x_{i_1} \ldots x_{i_k})}{\partial(u_1 \ldots u_k)} d\mathbf{u}$$

where D is the parameter domain of Φ and The functions $a_{i_1 \ldots i_k}$ are assumed to be real and continuous in E.

Products of basic k-forms and general forms

Suppose $I = \{i_1, \dots, i_p\}$ and $J = \{j_1, \dots, j_q\}$ where $1 \leq i_1, \dots, i_p \leq n$ and1 $\leq j_1, \dots, j_q \leq n$. The product of the corresponding basic forms dx_I and dx_J in $\mathbb{R}^n$ is a $(p+q)$-form in $\mathbb{R}^n$, denoted by the symbol $dx_I \wedge dx_J$, and de fined by

$$dx_I \wedge dx_J = dx_{i_1} \wedge \dots dx_{i_p} \wedge dx_{j_1} \wedge \dots dx_{j_q}$$

f I and J have an element i n common, then $dx_I \wedge dx_J = 0$.

Suppose ω and λ are p- and q-forms, respectively, in some open set $E \subseteq \mathbb{R}^n$, with standard presentations

$$\omega = \sum_I b_I(\mathbf{x}) dx_I, \quad \lambda = \sum_J c_J(\mathbf{x}) dx_J$$

where I and J range over all increasing p-indices and q-indices. Then

$$\omega \wedge \lambda = \sum_{I,J} b_I(\mathbf{x}) c_J(\mathbf{x}) dx_I \wedge dx_J$$

Differentiation of Forms

A differentiation operator d associates a $(k+1)$-form dw to each k-form ro of class $\mathcal{C}'$ (the class of continuously differentiable functions) in some open set $E \subseteq \mathbb{R}^n$.

A 0-form of class $\mathcal{C}'$ in E is just a real function $f \in \mathcal{C}'(E)$, and we define

$$d\omega = \sum_{i=1}^{n} (D_i f)(\mathbf{x}) dx_i$$

If $\omega = \sum_I b_I(\mathbf{x}) dx_I$ is the standard presentation of a k-form ro, and $b_I \in \mathcal{C}'(E)$ for each increasing k-index I, then we define

$$d\omega = \sum_I (db_I) \wedge dx_I$$

Differentiation of Forms

- If ω and λ are k- and m-forms, respectively, of class $\mathcal{C}'(E)$, then
$$d(\omega \wedge \lambda) = d\omega \wedge \lambda + (-1)^n \omega \wedge d\lambda$$
- If ω is of class $\mathcal{C}'(E)$, then $d^2\omega = d(d\omega) = 0$.