Advanced Calculus

Alternating k-Forms

ThinkBS: Basic Sciences in Engineering Education

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Alternating k-Forms

Let V be a real vector space. An alternating k-form ω on V is a map

$$\omega : V^k \to \mathbb{R}$$

that is multilinear (i.e. linear in each of the k variables) and has the additional property that $\omega(v_1, v_2, \ldots, v_k) = 0$ if $v_1, v_2, \ldots, v_k \in V$ are linearly dependent.

The vector space of alternating k-forms on V is denoted by $Alt^k V$. We define $Alt^0 V = \mathbb{R}$.

Alternating k-Forms

For multilinear maps $\omega : V^k \to W$, the following conditions are equivalent:

- ω is alternating; that is, $\omega(v_1, v_2, \ldots, v_k) = 0$ if $v_1, v_2, \ldots, v_k \in V$ are linearly dependent.
- $\omega(v_1, v_2, \ldots, v_k) = 0$ if any two of the v_i are equal, that is, if there are indices i, j with $i \neq j$ and $v_i = v_j$.
- Interchanging two of the variables switches the sign: $\omega(v_1, v_2, \ldots, v_k) = -\omega(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k)$ for $i < j$.

Alternating k-Forms

If $(e_1, e_2, \ldots, e_n)$ is a basis of V and ω an alternating k-form on V then the numbers

$$a_{\mu_1 \ldots \mu_k} = \omega(e_{\mu_1}, e_{\mu_2}, \ldots, e_{\mu_k})$$

for $1 \leq \mu_i \leq n$, are called the components of ω with respect to the basis.

If $(e_1, e_2, \ldots, e_n)$ is a basis of V, then the map

$$\begin{aligned} Alt^k V &\to \mathbb{R}^{\binom{n}{k}} \\ \omega &\mapsto (\omega(e_{\mu_1}, e_{\mu_2}, \ldots, e_{\mu_k}))_{\mu_1 < \ldots < \mu_k} \end{aligned}$$

is an isomorphism.

Let $p \in U$, where (U, h) is a chart with coordinates $x^1, x^2, \dots, x^n$, i.e. $h = (x^1, x^2, \dots, x^n)$. Then the μth vector of the basis of T_pM given by the coordinates will be denoted by $\frac{\partial}{\partial x^\mu} \in T_pM$ and abbreviated $\partial_\mu \in T_pM$.

We denote the component functions of a k-form ω on M relative to a chart (U, h) by

$$\omega_{\mu_1 \dots \mu_k} = \omega(\partial_{\mu_1}, \dots, \partial_{\mu_k}) : U \to \mathbb{R}$$

and call a k-form continuous, differentiable, etc. if its component functions relative to the charts of some atlas in the differentiable structure on M are continuous, differentiable, etc.