Advanced Calculus

Tangent Spaces

ThinkBS: Basic Sciences in Engineering Education

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Recall that locally at x, the linear approximation of a map $f : E \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is the differential $df_x : \mathbb{R}^n \to \mathbb{R}^m$ of f at x. The differential is characterized by $f(\mathbf{x} + \mathbf{h}) = f(\mathbf{x}) + df_x.\mathbf{h} + r(\mathbf{h})$, where $\lim_{h \to 0} \frac{|r(\mathbf{h})|}{|\mathbf{h}|} = 0$, (and given by the Jacobian matrix). But how can a differentiable map $f : M \to N$ between two manifolds be characterized locally at $p \in M$ by a linear map?

For this we need to consider the concept of tangent spaces.

Tangent Spaces in Euclidean Space

If $M \subseteq \mathbb{R}^N$ is an n-dimensional manifold, $p \in M$, and (U, h) is a chart on $\mathbb{R}^N$ around p that flattens M, then the vector subspace of $\mathbb{R}^N$ defined by

$$T_p^{sub} M = dh_p^{-1}(\mathbb{R}^N \times \{0\})$$

is independent of the choice of charts. It is called the (submanifold) tangent space of M at the point p.

Simply, we define the tangent space $T_p M$ (for the case of Euclidean setting $f : M \subseteq \mathbb{R}^n \to \mathbb{R}^m$) as the image of the map $df_p : \mathbb{R}^n \to \mathbb{R}^m$.