Advanced Calculus

Analytic and Infinitely Differentiable Functions

ThinkBS: Basic Sciences in Engineering Education

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Analytic Functions

A function $f : D \to \mathbb{C}$, $D \subseteq \mathbb{C}$ open, which is complex differentiable at every point of D is also called (complex) analytic or holomorphic or regular in D.

f is called analytic at a point $a \in D$ iff there exists an open neighborhood $U \subseteq D$ of a such that f is analytic in U.

Example: The function $f(z) = z\bar{z}$ is complex differentiable at $a = 0$, but is not analytic at 0. (why?)

Infinitely Differentiable Functions

Fact: If a function $f : D \to \mathbb{C}$ on $D \subseteq \mathbb{C}$ open, is analytic, then it is complex differentiable of any order and further, f is representable as a power series (in fact, its Taylor series) in each open disk fully contained in D.

For real functions this is not the case. We have seen examples of functions which are differentiable but their differential is not even continuous.

We say a function $f : \mathbb{R} \to \mathbb{R}$ is of class $\mathcal{C}^n$, if it has n-th continuous derivative. f is of class $\mathcal{C}^\infty$ if it is differentiable of all degrees. f is of class $\mathcal{C}^\omega$ if it can be represented as a power series around each point.

Infinitely Differentiable Functions

With the notation above, we can say that for real functions $f : \mathbb{R} \to \mathbb{R}$

$$\mathcal{C} \supset \mathcal{C}^2 \supset \mathcal{C}^3 \supset \cdots \supset \mathcal{C}^\infty \supset \mathcal{C}^\omega$$

and the inclusions are proper; there exists n times differentiable functions that do not have $(n+1)$-th derivative. (Can you construct some examples?)

There are also infinitely differentiable functions which cannot be represented as Taylor or power series. For example the following function $f : \mathbb{R} \to \mathbb{R}$ is such an example around 0:

$$f(x) = \begin{cases} e^{\left(-\frac{1}{x^2}\right)} & \textit{if} \quad x \neq 0 \\ 0 & \textit{if} \quad x = 0 \end{cases}$$

In contrast with the complex case, this is why we said before that the complex differentiability is a strong restriction.