Advanced Calculus

Cauchy-Riemann Equations and Implications

ThinkBS: Basic Sciences in Engineering Education

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Cauchy-Riemann Equations

For a function $f : \mathbb{C} \to \mathbb{C}$ we can split its real and imaginary parts and for $z = x + iy$ we have $f(z) = u(x, y) + iv(x, y)$ and we can consider f as a function from $\mathbb{R}^2$ to $\mathbb{R}^2$. In this case if the function f is totally differentiable at $a \in D \subseteq \mathbb{R}^2$ (D open), we know that then the partial derivatives of u and v exist at a and the Jacobian is given by

$$\begin{pmatrix} \frac{\partial u}{\partial x}(a) & \frac{\partial v}{\partial x}(a) \\ \frac{\partial u}{\partial y}(a) & \frac{\partial v}{\partial y}(a) \end{pmatrix}$$

Considering this and the Theorem above we can say that:

Cauchy-Riemann Equations

For a function $f : D \to \mathbb{C}$, $D \subseteq \mathbb{C}$ open, $a \in D$ the following two statements are equivalent:

- f is complex differentiable at a.
- f is totally differentiable at a in the sense of real analysis ($\mathbb{C} = \mathbb{R}^2$), and for $u := \mathfrak{Re}(f)$ and $v := \mathfrak{Im}(f)$ the following differential equations hold:

$$\frac{\partial u}{\partial x}(a) = \frac{\partial v}{\partial y}(a) \quad \textit{and} \quad \frac{\partial u}{\partial y}(a) = -\frac{\partial v}{\partial x}(a)$$

or simply as

$$u_x = v_y \quad \textit{and} \quad u_y = -v_x$$

In case that one of the above holds we have:

$$f'(a) = u_x(a) + iv_x(a) = v_y(a) - iu_y(a)$$

Complex Differentiation: Examples

Example: For $f(z) = z^2$, we have $f'(z) = 2z$.

Let's split f into real and imaginary parts and study the Cauchy-Riemann equations. We have $f(x+iy) = (x+iy)^2 = (x^2 - y^2) + i(2xy)$ and thus $u(x,y) = x^2 - y^2$ and $v(x,y) = 2xy$. We have:

$$u_x = 2x = v_y \quad and \quad u_y = -2y = -v_x$$

So we have $f'(x+iy) = 2x + i(2y) = 2(x+iy)$ which is consistent with what we calculated before.

Characterization of locally constant functions

Let $f : D \to \mathbb{C}$, $D \subseteq \mathbb{C}$ open, then the followings are equivalent:

- f is locally constant in D (in other words, f is constant in some neighborhood of any point of D).
- f is complex differentiable for all $z \in D$ and

$$f'(z) = 0 \quad \textit{for all } z \in D$$

Note that if f is a complex differentiable function that has only real values, it follows from the Cauchy-Riemann equations that the derivative of f vanishes, and thus the function f is locally constant. This shows that complex differentiability is a strong restriction.

Example: The functions $f(z) = |\sin z|$ and $g(z) = \mathfrak{Re}(z)$ are not complex differentiable in $\mathbb{C}$. (why?)