Advanced Calculus

Differentiation of Complex Functions

ThinkBS: Basic Sciences in Engineering Education

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Differentiation of Complex Functions

Let's review the definition of derivative for a function $f : E \subseteq \mathbb{R}^n \to \mathbb{R}^m$. Pick $\mathbf{x} \in E$. If there exists a linear transformation $L : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\lim_{h \to 0} \frac{|f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) - L\mathbf{h}|}{|\mathbf{h}|} = 0$$

then we say that f is differentiable at $\mathbf{x}$, and we write $f'(\mathbf{x}) = L$.

If we consider $\mathbb{C} = \mathbb{R} \times \mathbb{R}$ then there are much similarities between complex derivative and real derivative. The formulation is almost the same but one must remember that the divisions and norms are complex.

Take $D \subseteq \mathbb{C}$. A function $f : D \to \mathbb{C}$ is said to be (complex) differentiable at the point $a \in D$ iff the following limit exists:

$$\lim_{z \to a} \frac{f(z) - f(a)}{z - a}$$

We denote this (complex) value by $f'(a)$.

If f is differentiable at each point of D, then one can consider the complex derivative as a function on D:

$$\begin{aligned} f' : & D \to \mathbb{C} \\ & z \mapsto f'(z) \end{aligned}$$

In the special case that $D = [a, b] \subseteq \mathbb{R}$, we can write $f(x) = u(x) + iv(x)$. In this case $f'(x) = u'(x) + iv'(x)$.

Equivalent formulations of Complex Derivation

Assume that $a \in D \subseteq \mathbb{C}$ is an accumulation point, $f : D \to \mathbb{C}$ and $l \in \mathbb{C}$. Then the following statements are equivalent:

- f is complex differentiable at a, and there has the derivative l ($f'(a) = l$).
- There exists a function $\phi : D \to \mathbb{C}$ which is continuous at a such that

$$f(z) = f(a) + \phi(z)(z-a) \quad \textit{and} \quad \phi(a) = l$$

- There exists a function $\rho : D \to \mathbb{C}$ which is continuous at a such that

$$f(z) = f(a) + l(z-a) + \rho(z)(z-a) \quad \textit{and} \quad \rho(a) = 0$$

- If one defines $r : D \to \mathbb{C}$ by the equation $f(z) = f(a) + l(z-a) + r(z)$ then

$$\lim_{z \to a} \frac{r(z)}{z-a} = 0 \quad \textit{or equivalently} \quad \lim_{z \to a} \frac{r(z)}{|z-a|} = 0$$

Properties of Complex Derivative

Let the functions $f, g : D \subseteq \mathbb{C} \to \mathbb{C}$ be complex differentiable at $a \in D$ and $\lambda \in \mathbb{C}$. Then the functions:

- $f + g$ is complex differentiable at a, and $(f + g)'(a) = f'(a) + g'(a)$
- λf is complex differentiable at a, and $(\lambda f)'(a) = \lambda f'(a)$
- fg is complex differentiable at a, and $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$
- If $g(a) =\neq 0$, $\frac{f}{g}$, is complex differentiable at a, and $(\frac{f}{g})'(a) = \frac{f'(a)g(a)-f(a)g'(a)}{g^2(a)}$

Complex Derivative and Totally Differentiable Functions

As we have seen before, we call a function $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$ totally differentiable at a point $a \in D$ if there exists an $\mathbb{R}$-linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ such that such that

$$f(\mathbf{x}) = f(a) + L(\mathbf{x} - a) + r(\mathbf{x})$$

with $\lim_{\mathbf{x} \to a} \frac{r(\mathbf{x})}{|\mathbf{x} - a|} = 0$. Here, $|\mathbf{x} - a|$ denotes the Euclidean distance between $\mathbf{x}$ and a.

The linear map L is uniquely determined and is called the Jacobian of f at a or the total differential of f at a, or the tangent map to f at a.

Complex Derivative and Totally Differentiable Functions

For a function $f : D \to \mathbb{C}$, $D \subseteq \mathbb{C}$ open, $a \in D$, the following two statements are equivalent:

- f is complex differentiable at a.
- f is totally differentiable at a (in the sense of real analysis by considering $\mathbb{C} = \mathbb{R} \times \mathbb{R}$), and the Jacobian $L : \mathbb{C} \to \mathbb{C}$ is of the form

  $$L(z) = l.z$$

  with l a suitable complex number. Of course the number l is the derivative $f'(a) = l$.

Here a question rises: When is an R-linear map $L : \mathbb{R}^2 \to \mathbb{R}^2$ also $\mathbb{C}$-linear? In other words for which $\mathbb{R}$-linear maps $L : \mathbb{R}^2 \to \mathbb{R}^2$ exists a complex number $l \in \mathbb{C} = \mathbb{R}^2$ such that

$$L(z) = lz$$

Theorem: For an R-linear map $L : \mathbb{R}^2 \to \mathbb{R}^2$ the following four statements are equivalent:

- There exists a complex number l with $L(z) = lz$.
- L is $\mathbb{C}$-linear.
- $L(i) = iL(1)$.
- The matrix with respect to the canonical basis $1 = (1, 0)$ and $i = (0, 1)$ has the special form

$$\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \quad \alpha, \beta \in \mathbb{R}$$