Advanced Calculus

Dini's Theorem, Stone-Weierstrass Theorem

ThinkBS: Basic Sciences in Engineering Education

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Dini's Theorem

Let $K \subseteq \mathbb{R}$ be a compact set and consider $(f_n) \in \mathcal{F}(E, \mathbb{R})^{\mathbb{N}}$ to be a sequence of continuous functions converging pointwise to a continuous function $f : K \to \mathbb{R}$. If (f_n) is increasing (i.e., $f_n(x) \leq f_{n+1}(x)$ for all $x \in K$ and $n \in \mathbb{N}$) or decreasing, then f_n converges uniformly to f on K.

Dini's theorem can also be formulated for the series:

Let $K \subseteq \mathbb{R}$ be a compact set and consider $(f_n) \in \mathcal{F}(E, \mathbb{R})^{\mathbb{N}}$ to be a sequence of continuous functions such that the series $\sum f_n$ converges pointwise to a continuous sum $s : K \to \mathbb{R}$. If the f_n are all nonnegative on K (i.e., $f_n(x) \geq 0$) for all $x \in K$ and $n \in \mathbb{N}$, then $\sum f_n$ converges uniformly to s on K.

The Stone-Weierstrass Theorem

Assume that X is a metric space and denote the set of all complex valued, continuous, bounded functions with domain X by $\mathcal{C}(X)$. We define the supremum norm for each $f \in \mathcal{C}(X)$ as

$$||f|| = sup_{x \in X} \ |f(x)|$$

By defining $d(f, g) = ||f - g||$ we can turn $\mathcal{C}(X)$ into a metric space. It is clear that a sequence $(f_n) \in \mathcal{C}(X)^{\mathbb{N}}$ is convergent iff $(f_n) \in X^{\mathbb{N}}$ is uniformly convergent. (why?)

Stone-Weierstrass Theorem states that polynomials are a dense subset of $\mathcal{C}(X)$:

If f is a continuous complex function on $[a, b]$, there exists a sequence of polynomials P_n such that P_n converges uniformly to f on $[a, b]$. If f is real, the P_n may be taken real.

Continuous Nowhere Differentiable Function

Consider the sawtooth function:

$$f_0(x) = \begin{cases} x - \lfloor x \rfloor & \textit{if} \quad x \leq \lfloor x \rfloor + \frac{1}{2} \\ \lfloor x \rfloor + 1 - x & \textit{if} \quad x > \lfloor x \rfloor + \frac{1}{2} \end{cases}$$

Then $f_0(x)$ is the distance from x to the nearest integer, i.e., $f_0(x) = d(x, \mathbb{Z})$, and is a continuous, periodic function on R with period 1.

Define $f_n(x) = 4^{-n} f_0(4^n x)$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}_0$. Then f_n is also a continuous sawtooth function (with period $\frac{1}{4^n}$), whose graph consists of line segments of slope 1 or -1. Since $0 \leq f_0(x) \leq \frac{1}{2}$ we have $0 \leq f_n(x) \leq \frac{1}{2.4^n}$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}_0$.

Continuous Nowhere Differentiable Function

Now consider $f(x) = \sum_{n=0}^{\infty} f_n(x)$ for all $x \in \mathbb{R}$. From Weierstrass M-Test it follows that the series $\sum_{n=0}^{\infty} f_n(x)$ uniformly converges to f and

Theorem (Van der Waerden): The function $f : \mathbb{R} \to \mathbb{R}$ defined as above is continuous but is not differentiable at any point in $\mathbb{R}$.