Advanced Calculus

Uniform Convergence

ThinkBS: Basic Sciences in Engineering Education

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Uniform Convergence

Let $E \subseteq \mathbb{R}$. We say $(f_n) \in \mathcal{F}(E, \mathbb{R})^{\mathbb{N}}$ converges uniformly on $E_0 \subseteq E$ to the function $f : E_0 \to \mathbb{R}$ if for every $\varepsilon > 0$ there is an integer N such that $n > N$ implies

$$|f_n(x) - f(x)| < \varepsilon$$

for all $x \in E_0$.

We should note that the existence of N in pointwise convergence depends on both ε and x but in uniform convergence it does not depend on x.

We show uniform convergence by $f_n \xrightarrow{u} f$.

It can be shown that

$$f_n \xrightarrow{u} f \quad iff \quad \lim_{n\to\infty} sup \, \{|f_n(x) - f(x)| \; : \; x \in E_0\} = 0$$

Let $E \subseteq \mathbb{R}$ and consider $(f_n) \in \mathcal{F}(E, \mathbb{R})^{\mathbb{N}}$. Then (f_n) converges uniformly on E to some function say f, if and only if for every $\varepsilon > 0$ there is an integer N such that $n > N$, $m > N$ and $x \in E$ implies

$$|f_n(x) - f_m(x)| < \varepsilon$$

.

We say that the series $\sum f_n(x)$ converges uniformly on E if the sequence of partial sums defined by $s_i(x) = \sum_{i=1}^{n} f_i(x)$ converges uniformly on E.

Suppose (f_n) is a sequence of functions defined on E, and suppose $|f_n(x)| \leq M_n$ for all $n \in \mathbb{N}$ and $x \in E$. If $\sum M_n$ converges, then $\sum_{i=1}^{n} f_i(x)$ converges uniformly on E.

Uniform Convergence: Examples

Example 1: Consider $f_n(x) = x^n \in \mathcal{F}([0,1], \mathbb{R})^{\mathbb{N}}$. We know the pointwise limit and want to see if the convergence is uniform or not. The answer is negative on $[0,1]$ because

$$\lim_{n\to\infty} sup \ \{|x^n - f(x)| \ : \ x \in [0,1]\} = \lim_{n\to\infty} 1 = 1 \neq 0$$

What can be said for a subset that does not include 1?

Example 2: Consider $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$ for all $x \in \mathbb{R}$. Then $f(x) = |x|$ and the convergence is in fact uniform since

$$sup \ \{|\sqrt{x^2 + \frac{1}{n}} - |x|| \ : \ x \in \mathbb{R}\} = sup \ \{\frac{\frac{1}{n}}{\sqrt{x^2 + \frac{1}{n}} + |x|} \ : \ x \in \mathbb{R}\}$$

$$= \frac{\frac{1}{n}}{\frac{1}{\sqrt{n}}} = \frac{1}{\sqrt{n}} \to 0$$

Uniform Convergence: Examples

Example 3: Consider $f_n(x) = \frac{\sin(n^2x)}{n}$ for all $x \in \mathbb{R}$ with pointwise limit $f(x) = 0$. The convergence is also uniform on $\mathbb{R}$. To see this note that $|\frac{\sin(n^2x)}{n}| \leq \frac{1}{n}$.

Example 4: Consider $f_n(x) = \frac{x}{1+nx^2}$ for all $x \in \mathbb{R}$. Then $f_n \xrightarrow{u} 0$. (why?)

Example 5: Consider $f_n(x) = \sqrt{n}x^n(1-x)$ for all $x \in [0,1]$. Then $f_n \xrightarrow{u} 0$. (why?)

Uniform Convergence Properties

Let $c \in \mathbb{R}$ and consider (f_n) and (g_n) are two uniformly convergent functions on E to f and g respectively. Then

- $(f_n + g_n) \xrightarrow{u} (f + g)$ on E.
- $cf_n \xrightarrow{u} cf$ on E.
- If for all n, (f_n) and (g_n) are bounded then $f_n g_n \xrightarrow{u} fg$ on E.
- If there is an $m > 0$ such that for all n and x, $|f_n(x)| \geq m$ (in this case we call (f_n) uniformly far from zero), then $\frac{1}{f_n} \xrightarrow{u} \frac{1}{f}$ on E.
- If for all n, $f_n : E \to [a, b]$ and if $\phi : [a, b] \to \mathbb{R}$ is uniformly continuous then $\phi(f_n) \xrightarrow{u} \phi(f)$ on E.

Uniform Convergence and Continuity

Let $E_0 \subseteq E \subseteq \mathbb{R}$ and $(f_n) \in \mathcal{F}(E, \mathbb{R})^{\mathbb{N}}$. If each f_n is continuous at some $x_0 \in E_0$ and $f_n \xrightarrow{u} f$ on E_0 then f is also continuous at x_0. Thus, if each f_n is continuous on E_0, then so is the limit function f.

Corollary: Let $E_0 \subseteq E \subseteq \mathbb{R}$ and and $(f_n) \in \mathcal{F}(E, \mathbb{R})^{\mathbb{N}}$. If each f_n is continuous at some $x_0 \in E_0$ and the series $\sum f_n$ converges uniformly on E_0 to a sum $s \in \mathcal{F}(E_0, \mathbb{R})$ then s is also continuous at x_0. In particular, if each f_n is continuous on E_0 then so is the sum s.

Uniform Convergence and Integrability

Let (f_n) be a sequence of Riemann integrable functions on $[a, b]$. If $f_n \xrightarrow{u} f$ on $[a, b]$ then f is also Riemann integrable on $[a, b]$ and for all $x \in [a, b]$ we have

$$\int_a^x f(t)dt = \lim_{n\to\infty} \int_a^x f_n(t)dt$$

Corollary (Term-by-Term Integration): Let (f_n) be a sequence of Riemann integrable functions on $[a, b]$. If $\sum f_n$ converges uniformly to a sum s on $[a, b]$ then s is also Riemann integrable on $[a, b]$ and for all $x \in [a, b]$ we have

$$\int_a^x s(t)dt = \sum_{n=1}^{\infty} \int_a^x f_n(t)dt$$

Uniform Convergence and Differentiability

Let (f_n) be a sequence of differentiable functions on $[a, b]$ such that $f_n(x_0)$ converges for some $x_0 \in [a, b]$. If the sequence (f_n') of derivatives converges to a function g uniformly on $[a, b]$ then the sequence (f_n) converges uniformly on $[a, b]$ to a differentiable function f, and for all $x \in [a, b]$ we have

$$f'(x) = \lim_{n \to \infty} f_n'(x) = g(x)$$

Corollary (Term-by-Term Differentiation): Let (f_n) be a sequence of differentiable functions on $[a, b]$ such that the series $\sum f_n(x_0)$ converges for some $x_0 \in [a, b]$. If the series $\sum f_n'$ of derivatives converges uniformly on $[a, b]$ then the series $\sum f_n$ converges uniformly on $[a, b]$ to a differentiable sum s and for all $x \in [a, b]$ we have

$$s'(x) = (\sum_{n=1}^{\infty} f_n(x))' = \sum_{n=1}^{\infty} f_n'(x)$$