Advanced Calculus

Sequences and Series of Functions, Pointwise Convergence

ThinkBS: Basic Sciences in Engineering Education

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Up to this point, we have discussed numerical sequences and series. We have also studied the radius of converges for the series for which we introduced a variable and using this we have been able to define new functions like sine, cosine and exponential function.

In what follows, instead of putting numbers in a 'sequence', we will put functions in a sequence and we will construct series that 'add' functions instead of numbers and will study properties of such sequences and series.

Given a set $E \subseteq \mathbb{R}$, the set of all functions from E to $\mathbb{R}$ is denoted by $\mathcal{F}(E, \mathbb{R})$. Every sequence $(f_n) \in \mathcal{F}(E, \mathbb{R})^{\mathbb{N}}$ is called a sequence of functions.

For each such sequence if we choose $x \in E$ we can assign a numerical sequence $(f_n(x)) \in \mathbb{R}^{\mathbb{N}}$.

This numerical sequence may or may not converge. We first study the convergence of these numerical sequences and later will study the sequence itself, as a sequence of functions.

Likewise we can also talk about the series of functions $\sum_{n=1}^{\infty} f_n$.

Pointwise Convergence

For each $(f_n) \in \mathcal{F}(E, \mathbb{R})^{\mathbb{N}}$ let $E_0 \subseteq E$ be the set of all points $x \in E$ such that the numerical sequence $(f_n(x)) \in \mathbb{R}^{\mathbb{N}}$ converges and let

$$f(x) = \lim_{n \to \infty} f_n(x) = \lim f_n(x) \quad \textit{for all } x \in E_0$$

In this case the sequence $(f_n) \in \mathcal{F}(E, \mathbb{R})^{\mathbb{N}}$ is said to be pointwise convergent on E_0 and the function f defined on E_0 is called the pointwise limit of the sequence (f_n) on E_0.

One should note that the existence of the limit here depends on points of E and for each different point of E we calculate the limit separately.

Pointwise Convergence: Examples

Example 1: Consider $f_n(x) = x^n \in \mathcal{F}([0,1], \mathbb{R})^{\mathbb{N}}$. The pointwise limit is equal to

$$f(x) = \lim f_n(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

Note that while for all n, f_n's are continuous but the pointwise limit is not.

Example 2: Consider $f_n(x) = \sqrt{x^2 + \frac{1}{n}}$ for all $x \in \mathbb{R}$. Then

$$f(x) = \lim f_n(x) = |x|$$

Note that all $f_n(x)$'s are differentiable but the pointwise limit is not.

Pointwise Convergence: Examples

Example 3: Consider $f_n(x) = \frac{\sin(n^2 x)}{n}$ for all $x \in \mathbb{R}$. Then

$$f(x) = \lim f_n(x) = 0$$

Here since $f(x) = 0$ we also have $f'(x) = 0$. But for each n, $f_n'(x) = n\cos(n^2 x)$ and $\lim_{n\to\infty} f_n'(x)$ does not exist for all $x \in \mathbb{R}$.

Example 4: Consider $f_n(x) = \lfloor \cos^2(n!\pi x) \rfloor$ for all $x \in [0,1]$. Then

$$f(x) = \lim f_n(x) = \begin{cases} 1 & \textit{if } x \in [0,1] \cap \mathbb{Q} \\ 0 & \textit{if } x \in [0,1] \setminus \mathbb{Q} \end{cases}$$

Here the pointwise limit is not Riemann integrable, $f \notin \mathcal{R}$ but for all n, $f_n \in \mathcal{R}$ (why?)

Pointwise Convergence: Examples

Example 5: Let $f_n(x) = nx(1-x^2)^n$ for all $x \in [0,1]$. We have $f(x) = \lim f_n(x) = 0$. Thus

$$\int_0^1 f(x)dx = \int_0^1 \lim f_n(x)dx = \int_0^1 0dx = 0$$

On the other hand we have

$$\lim_{n\to\infty}\left(\int_0^1 f_n(x)dx\right) = \lim_{n\to\infty}\left(\int_0^1 nx(1-x^2)^n dx\right) = \lim_{n\to\infty}\frac{n}{2(n+1)} = \frac{1}{2}$$

Pointwise Convergence: Examples

Example 6: Let $f_n(x) = \frac{x^2}{(1+x^2)^n}$ for all $x \in [0,1]$. Consider

$$f(x) = \sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \frac{x^2}{(1+x^2)^n}$$

Then we have

$$f(x) = \lim f_n(x) = \begin{cases} 0 & if \;\; x = 0 \\ 1 + x^2 & if \;\; x \neq 0 \end{cases}$$

The Moral of Examples

- Even if all the functions f_n are continuous at a point x_0 the limit f may be discontinuous there.
- Even if all the f_n are differentiable at x_0, f need not be differentiable at x_0. Even if $f'(x_0)$ exists, the sequence of derivatives $f_n'(x_0)$ need not converge to $f'(x_0)$.
- Even if all the f_n are Riemann integrable on some interval $[a, b]$ the limit f need not be integrable on $[a, b]$. Even if $\int_a^b f(x)dx$ exists, it need not be the limit of the sequence of integrals $\int_a^b f_n(x)dx$.

What we would like to have is a notion of converges that behaves well with the limit, continuity, differentiation and integration. For this we need to replace the notion of pointwise convergence with uniform convergence.