Advanced Calculus

Riemann and Darboux Integrals: Theorems

ThinkBS: Basic Sciences in Engineering Education

Kadir Has University, Turkey

Theorems About Existence of Integral

- If f is continuous on $[a, b]$ then $f \in \mathcal{R}(\alpha)$ on $[a, b]$.
- If f is monotonic on $[a, b]$, and if α is continuous (and monotonic) on $[a, b]$, then $f \in \mathcal{R}(\alpha)$ on $[a, b]$.
- Suppose f is bounded on $[a, b]$, f has only finitely many points of discontinuity on $[a, b]$, and α is continuous at every point at which f is discontinuous. Then $f \in \mathcal{R}(\alpha)$ on $[a, b]$.
- If $f \in \mathcal{R}$ on $[a, b]$ then f is bounded on $[a, b]$.
- Suppose $f \in \mathcal{R}(\alpha)$ on $[a, b]$, $m < f < M$, g is continuous on $[m, M]$, and $h(x) = g(f(x))$ on $[a, b]$. Then $h \in \mathcal{R}(\alpha)$ on $[a, b]$.
- If $f \in \mathcal{R}(\alpha)$ then $|f| \in \mathcal{R}(\alpha)$, but the converse is not true. (why?) Also $|\int_a^b f d\alpha| \leq \int_a^b |f| d\alpha$.

Properties of Integral

- If f and $g \in \mathcal{R}(\alpha)$ on $[a, b]$ and $c \in \mathbb{R}$, then $f + g \in \mathcal{R}(\alpha)$ and $cf \in \mathcal{R}(\alpha)$. Also

$$\int_a^b (f+g)d\alpha = \int_a^b fd\alpha + \int_a^b gd\alpha, \quad \int_a^b cfd\alpha = c\int_a^b fd\alpha$$

- If $f(x) \leq g(x)$ on $[a, b]$ and $f, g \in \mathcal{R}(\alpha)$ then

$$\int_a^b fd\alpha \leq \int_a^b gd\alpha$$

- If $f \in \mathcal{R}(\alpha)$ on $[a, b]$ and $a < c < b$ then $f \in \mathcal{R}(\alpha)$ on $[a, c]$ and $[c, b]$ and

$$\int_a^b fd\alpha = \int_a^c fd\alpha + \int_c^b fd\alpha$$

Properties of Integral

- If $f \in \mathcal{R}(\alpha)$ on $[a, b]$ and if $|f(x)| \leq M$ on $[a, b]$ then

$$|\int_a^b f d\alpha| \leq M(\alpha(b) - \alpha(a))$$

- If f and $g \in \mathcal{R}(\alpha)$ on $[a, b]$ then $fg \in \mathcal{R}(\alpha)$ on $[a, b]$ and also Cauchy–Bunyakovsky–Schwarz inequality for integrals holds:

$$(\int_a^b fg d\alpha)^2 \leq (\int_a^b f^2 d\alpha)(\int_a^b g^2 d\alpha)$$

- If $f \in \mathcal{R}(\alpha_1)$ and $f \in \mathcal{R}(\alpha_2)$ on $[a, b]$ and $c \in \mathbb{R}^{>0}$, then $f \in \mathcal{R}(\alpha_1 + \alpha_2)$ and $f \in \mathcal{R}(c\alpha_1)$ and we have

$$\int_a^b f d(\alpha_1 + \alpha_2) = \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2, \quad \int_a^b f d(c\alpha_1) = c \int_a^b f d\alpha_1$$

We have said that integration is in fact a continuous infinite summation, so this means it is a generalization of infinite sums. To see this claim define unit step function I as following:

$$I(x) = \begin{cases} 0 \; if \;\; x \leq 0 \\ 1 \; if \;\; x > 0 \end{cases}$$

Then if $a < s < b$, f is bounded on $[a, b]$, f is continuous at s, and $\alpha(x) = I(x - s)$, then

$$\int_a^b f d\alpha = f(s)$$

Use of Integrals as Series

Now Suppose $\{c_n\}_{n=1}^{\infty}$ is a **non-negative** sequence $(c_n \geq 0)$, $\sum c_n$ converges, $\{s_n\}_{n=1}^{\infty}$ is a sequence of distinct points in (a, b), and

$$\alpha(x) = \sum_{n=1}^{\infty} c_n I(x - s_n)$$

Let f be continuous on $[a, b]$. Then

$$\int_a^b f d\alpha = \sum_{n=1}^{\infty} c_n f(s_n)$$

The Relation Between Darboux and Riemann Integral

Assume α increases monotonically and $\alpha' \in \mathcal{R}$ on $[a, b]$. Let f be a bounded real function on $[a, b]$.
Then $f \in \mathcal{R}(\alpha)$ if and only if $f\alpha' \in \mathcal{R}$. In that case

$$\int_a^b f d\alpha = \int_a^b f(x)\alpha'(x)dx$$

If α is a step function, the integral reduces to a finite or infinite series. If α has an integrable derivative, the integral reduces to an ordinary Riemann integral . This makes it possible in many cases to study series and integrals simultaneously.

Change of Variable

Suppose ϕ is a strictly increasing continuous function that maps an interval $[A, B]$ onto $[a, b]$. Suppose α is monotonically increasing on $[a, b]$ and $f \in \mathcal{R}(\alpha)$ on $[a, b]$. Define β and g on $[A, B]$ by

$$\beta(y) = \alpha(\phi(y)), \quad g(y) = f(\phi(y))$$

Then $g \in \mathcal{R}(\beta)$ and

$$\int_A^B g d\beta = \int_a^b f d\alpha$$

Integration and Differentiation

Let $f \in \mathcal{R}(\alpha)$ on $[a, b]$. For $a \leq x \leq b$ put

$$F(x) = \int_a^x f(t)dt$$

Then F is uniformly continuous on $[a, b]$; furthermore, if f is continuous at a point $x_0 \in [a, b]$, then F is differentiable at x_0 , and

$$F'(x_0) = f(x_0)$$

The Fundamental Theorem of Calculus

Let $f \in \mathcal{R}(\alpha)$ on $[a, b]$ and if there is a differentiable function F on $[a, b]$ such that $F' = f$ (F is called the anti-derivative of f), then

$$\int_a^b f(x)dx = F(b) - F(a) = [F(x)]_a^b$$

This theorem connects two concepts: On one hand we have a infinite (continuous) summation and on the other hand we have an operation which is inverse of taking derivative (anti-derivative!) At first it may seem they are completely irrelevant subjects but due to Fundamental Theorem they are quite related!

Integration by Parts

Suppose F and G are differentiable functions on $[a, b]$, $F' = f \in \mathcal{R}$, $G' = g \in \mathcal{R}$. Then

$$\int_a^b F(x)g(x)dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)dx$$

Examples

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Example 2: Define $\alpha(x) = x^2$. Then $\int_1^2 \frac{1}{x} d\alpha = \int_1^2 \frac{1}{x}(2x)dx = 2$.

Example 3: Define $\alpha(x) = x$. Then

$\int_0^{\frac{\pi}{2}} (\sin x)^2 \cos x dx = \int_0^1 u^2 du = [\frac{u^3}{3}]_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$.

Here we used both change of variable and Fundamental theorem.

Example 4: First we define $\int_a^b f = -\int_b^a f$ and $\int_a^a f = 0$.

Consider $x > 0$. We define

$$\log x = \int_1^x \frac{1}{t} dt$$

Here $\log x$ is a continuous function on $(0, \infty)$ (why?) and also

$$(\log x)' = \frac{1}{x}$$

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