Advanced Calculus

Riemann and Darboux Integrals: Definitions

ThinkBS: Basic Sciences in Engineering Education

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Riemann and Darboux Sums

Let $f : [a, b] \to \mathbb{R}$ be a bounded function and consider a tagged partition $(\mathcal{P}, \tau) \in \mathbf{P}([a, b])$. The Riemann sum of f corresponding to this partition is defined as

$$S(f, \mathcal{P}, \tau) = \sum_{i=1}^{n} f(t_i)\Delta x_i$$

with $\Delta x_i = x_i - x_{i-1}$ for $i = 1, 2, \dots, n$.

Likewise lower and upper Darboux sums are respectively defined as $L(f, \mathcal{P}) = \sum_{i=1}^{n} m_i \Delta x_i$ and $U(f, \mathcal{P}) = \sum_{i=1}^{n} M_i \Delta x_i$.

Given that $m = inf_{x \in [a,b]} f(x)$ and $M = sup_{x \in [a,b]} f(x)$, we always have:

$$m(b-a) \leq L(f, \mathcal{P}) \leq S(f, \mathcal{P}, \tau) \leq U(f, \mathcal{P}) \leq M(b-a)$$

Darboux Sums and Refinements

Let $f : [a, b] \to \mathbb{R}$ be bounded and let $\mathcal{P}_1$ and $\mathcal{P}_2$ be two partitions of $[a, b]$. If $\mathcal{P}_2$ is a refinement of $\mathcal{P}_1$ ($\mathcal{P}_1 \subseteq \mathcal{P}_2$) then we have:

$$L(f, \mathcal{P}_1) \leq L(f, \mathcal{P}_2) \leq U(f, \mathcal{P}_2) \leq U(f, \mathcal{P}_1)$$

By considering a common refinement for two **arbitrary** partition $\mathcal{P}$ and $\mathcal{P}'$ and using the inequality stated above one can also see that:

$$L(f, \mathcal{P}) \leq U(f, \mathcal{P}')$$

Do you see how we use the step functions l, u and ϕ and the area under their graphs in order to give meaning to S, L and U?

Lower and Upper Darboux Integrals

For a bounded function $f : [a, b] \to \mathbb{R}$ the lower and upper Darboux integrals of f denoted $\underline{\int} f$ and $\overline{\int} f$ respectively, are defined to be the real numbers

$$\underline{\int} f = \underline{\int_a^b} f = \underline{\int_a^b} f(x)dx = sup\{L(f, \mathcal{P}) \mid \mathcal{P} \in \mathbf{P}([a, b])\}$$

and

$$\overline{\int} f = \overline{\int_a^b} f = \overline{\int_a^b} f(x)dx = inf\{U(f, \mathcal{P}) \mid \mathcal{P} \in \mathbf{P}([a, b])\}$$

It is clear that

$$\underline{\int_a^b} f \leq \overline{\int_a^b} f$$

Darboux Integrals: Examples

Example 1: For a constant function $f : [a, b] \to \mathbb{R}$ given by $f(x) = c$, $\underline{\int_a^b} f = \overline{\int_a^b} f = c(b - a)$ (why?)

Example 2: Consider the Dirichlet function $f(x) = \begin{cases} 1 \ \textit{if} \ \ x \in [a, b] \cap \mathbb{Q} \\ 0 \ \textit{if} \ \ x \in [a, b] \setminus \mathbb{Q} \end{cases}$. Since $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are both dense in $\mathbb{R}$, for every partition of $[a, b]$ we can tag this partition by both rational and irrational numbers. Also for each sub-interval of any partition we have $m_i = 0$ and $M_i = 1$. Thus $\underline{\int_a^b} f = 0$ and $\overline{\int_a^b} f = b - a$. (why?) Here $\underline{\int_a^b} f < \overline{\int_a^b} f$.

Riemann Integrable Functions

We call the function $f : [a, b] \to \mathbb{R}$ Riemann integrable, shown by $f \in \mathcal{R}$, if $\underline{\int_a^b} f = \overline{\int_a^b} f$. We show this common value by $\int_a^b f$ or by $\int_a^b f(x)dx$.

Here the value of integral only depends on f, a and b and the variable x is a dummy variable, and like index of summation for series can be replaces by any other dummy variable. So $\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(s)ds = \dots$

As we have seen in the examples above, constant functions are integrable but Dirichlet function is an example of non-integrable function.

Riemann-Stieltjes integral

Let α be a monotonically increasing function on $[a, b]$ (and thus bounded). Corresponding to each partition $\mathcal{P}$ of $[a, b]$, we write

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}).$$

Clearly $\Delta\alpha_i \geq 0$. Now for any bounded function $f : [a, b] \to \mathbb{R}$ define

$$L(f, \mathcal{P}, \alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i, \quad U(f, \mathcal{P}, \alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i$$

And put

$$\underline{\int_a^b} f d\alpha = sup\ L(f, \mathcal{P}, \alpha), \quad \overline{\int_a^b} f d\alpha = inf\ \ U(f, \mathcal{P}, \alpha)$$

Where sup and inf is taken over all partitions $\mathcal{P} \in \mathbf{P}([a, b])$.

Riemann-Stieltjes integral

One can show very similar properties for $\underline{\int_a^b} f d\alpha$ and $\overline{\int_a^b} f d\alpha$ holds as before.

If for a function $f : [a, b] \to \mathbb{R}$ we have $\underline{\int_a^b} f d\alpha = \overline{\int_a^b} f d\alpha = \int_a^b f d\alpha$ then we call it a Riemann-Stieltjes integrable function and show it by $f \in \mathcal{R}(\alpha)$.

Riemann's Criterion: $f \in \mathcal{R}(\alpha)$ on $[a, b]$ if and only if for every $\varepsilon > 0$ there exists a partition $\mathcal{P}$ such that

$$U(f, \mathcal{P}, \alpha) - L(f, \mathcal{P}, \alpha) < \varepsilon$$