Advanced Calculus

Introduction to Integral Calculus

ThinkBS: Basic Sciences in Engineering Education

Kadir Has University, Turkey

Step Functions and Partitions

A function $f : [a, b] \to \mathbb{R}$ is called a step function if we can partition its domain $[a, b]$ in such a way that the function is constant on each partition.

By a partition of the interval $[a, b]$ we mean a finite sequence $\mathcal{P} = (x_k)_{k=0}^{n}$ such that $x_0 = a < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$.

By a refinement $\mathcal{P}'$ of partition $\mathcal{P}$, we mean $\mathcal{P}$ plus some additional points from the interval. If $\mathcal{P}'$ is a refinement of $\mathcal{P}$ then clearly $\mathcal{P} \subseteq \mathcal{P}'$.

If we have two partitions $\mathcal{P}_1$ and $\mathcal{P}_2$ of the interval $[a, b]$, then clearly $\mathcal{P}_1 \cup \mathcal{P}_2$ is a common refinement of both partitions.

The set of all partitions of an interval $[a, b]$ is shown by $\mathbf{P}([a, b])$.

Tagged Partitions

Consider $\mathcal{P} = \{x_0, x_1, \ldots, x_n\} \in \mathbf{P}([a, b])$ and select an element $\tau = t_i$ from each sub-interval $[x_{i-1}, x_i]$. Then the partition $\mathcal{P}$ together with the sequence $(t_i)_{i=1}^{n}$ is called a tagged partition and is shown by $(\mathcal{P}, \tau)$.

Now consider $f : [a, b] \to \mathbb{R}$ be a bounded function and given a tagged partition $(\mathcal{P}, \tau) \in \mathbf{P}([a, b])$, we are going to construct the following step functions:

For all $x \in [a, b]$ define $\phi(x) = f(t_i)$ if $x \in [x_{i-1}, x_i]$.

This is clearly a step function based on $(\mathcal{P}, \tau)$.

Tagged Partitions and Step Functions

Now define $m_i = inf\{f(x) \mid x \in [x_{i-1}, x_i]\}$ and
$M_i = sup\{f(x) \mid x \in [x_{i-1}, x_i]\}$.

Define $l(x) = m_i$ if $x \in [x_{i-1}, x_i]$ and $u(x) = M_i$ if $x \in [x_{i-1}, x_i]$.

It is obvious that the function f is sandwiched between l (from below) and u from above and the function ϕ floats in between, sometimes below f and sometime above f.

What happens if we make our partition $(\mathcal{P}, \tau)$ gets more refined? If our function is "good enough", the more we add points to our partition, these step functions tend to completely come to an overlap with our function f. But for this to happen, from a finite sequence we need to go to an infinite sequence, not only countable but a partition with a cardinality equal to that of an interval: $\mathfrak{c}$.

Summation on continuum

We have discussed the notions of $\sum_{i=1}^{n}$ and $\sum_{i=1}^{\infty}$. The first one happens in a finite set-up. The later is happening in a infinite but countable set-up. How we can extend this summation to an uncountable (continuum) infinite one?

Why we need this at all? Consider a wire for which you want to calculate its weight and you know the density of wire at each point. Isn't the weight of the wire the sum of density of all points on the wire? But how many points are there on the wire? Exactly! Infinitely many!

Summation on continuum

Now put this wire on the interval $[a, b]$, and let the y-axis show the density at each point. First we will approximate the weight by the means of step functions defined above (The role that these step functions play, is to turn the problem to a set-up that we know (how?))

Isn't calculating the weight by the density function very similar to finding the area under the curve of the density function?

An infinite continuous summation of densities on a wire can geometrically be interpreted as putting together of infinitely many line segments of area zero glued continuously together to cover the area under the curve, to create an area!

By passing to limit, we will try to grasp an idea on how to do a continuous infinite summation, this is called integration!