Advanced Calculus

Differentiation Rules

ThinkBS: Basic Sciences in Engineering Education

Kadir Has University, Turkey

Differentiation Rules

Assume that $c \in \mathbb{R}$ and f and g are real-valued functions defined on an interval I and both are differentiable, then the following functions are differentiable and we have:

- $(f+g)' = f' + g'$
- $(cf)' = cf'$
- $(fg)' = f'g + g'f$
- $(\frac{f}{g})' = \frac{f'g - g'f}{g^2}$ where $g \neq 0$.

Assume $f : I \to \mathbb{R}$, $g : J \to \mathbb{R}$ and $f(I) \subset J$, if f is differentiable at $x \in I$ and g is differentiable at $f(x) \in J$, then according to **the chain rule**, the composite function gof is differentiable at x and we have

$$(gof)'(x) = g'(f(x)).f'(x)$$

Differentiation Rules

- Let $I \neq \emptyset$ be an open interval and let $f : I \to \mathbb{R}$ be an injective, continuous function. If f is differentiable at $x_0 \in I$ and $f'(x_0) \neq 0$ then the inverse function f^{-1} is differentiable at $y_0 = f(x_0)$ and we have

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))}$$

- Assume that $f : I \to \mathbb{R}$ is differentiable at point $x \in I$, then it can be shown that f must be continuous at x. Is the converse also true?
- If $f : [a, b] \to \mathbb{R}$ and $f'(x) = 0$ for all $x \in (a, b)$, then f is a constant function on $[a, b]$. Can you geometrically interpret this result?

Use of Differentiation Rules

Assume $f(x) = \begin{cases} x\sin(\frac{1}{x}) & if \ \ x \neq 0 \\ 0 & if \ \ x = 0 \end{cases}$ Show that for all $x \neq 0$ we have:

$$f'(x) = \sin(\frac{1}{x}) - \frac{1}{x}\cos(\frac{1}{x})$$

but f is not differentiable at $x = 0$.

This is another example of a function which is continuous everywhere and is differentiable everywhere except one point. Here f' is also continuous everywhere except $x = 0$.

Assume $f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & if \;\; x \neq 0 \\ 0 & if \;\; x = 0 \end{cases}$ Show that for all $x \neq 0$ we have:

$$f'(x) = 2x \sin(\frac{1}{x}) - \cos(\frac{1}{x})$$

and for $x = 0$ we have

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^2 \sin(\frac{1}{x})}{x} = 0$$

.

Hence f is differentiable for all $x \in \mathbb{R}$. We should note that the derivative function f' is not continuous at $x = 0$.

Use of Differentiation Rules

The function $\log : (0, \infty) \to \mathbb{R}$ (Natural Logarithm) is the inverse function of $f(x) = e^x$. Since for all $x \in \mathbb{R}$, $(e^x)' = e^x$ and $e^x > 0$, we can calculate the derivative of log on its domain as:

$$(\log x)' = \frac{1}{exp'(\log x)} = \frac{1}{exp(\log x)} = \frac{1}{x}$$

Similarly, we have $(\log |x|)' = \frac{1}{x}$ for all $x \in \mathbb{R} \setminus \{0\}$.

One can also use this result and the chain rule to show that if $u : I \to \mathbb{R}$ is differentiable and u is not zero on I, then $\log(|u(x)|)$ is differentiable on I and:

$$(\log |u(x)|)' = \frac{u'(x)}{u(x)}$$