Advanced Calculus

Equivalent Definition of Derivative and Geometric Interpretation

ThinkBS: Basic Sciences in Engineering Education

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Reformulation of the Definition

Suppose E is an open set in $\mathbb{R}$, $f : E \subseteq \mathbb{R} \to \mathbb{R}$, and $x \in E$. We can reformulate the definition of derivative:

Instead of $f(x+h) = f(x) + \alpha h + r(h)$ where $\lim_{h\to 0} |\frac{r(h)}{h}| = 0$, we can write α as $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$. Using this we can give the equivalent (and conventional!) definition of derivative as below:

If the limit $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ exists at point x, we say that the function f is differentiable at x and this value is called the derivative of f and is shown by $f'(x)$.

If f is differentiable on whole E, then f is said to be differentiable on E. In this case as x runs through E, $f'(x)$ is a function of x. We refer to this function as the derivative function.

Reformulation of the Definition

If we assume that $x_2 = x + h$ and $x_1 = x$, then $h = x_2 - x_1$ and we can rewrite the definition of derivative at point $x_1 = x$ as:

$$f'(x) = f'(x_1) = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

As one can see, all of these definitions and formulations are equivalent and we can use them as we desire.

Geometric Interpretations

The term $\frac{f(x_2)-f(x_1)}{x_2-x_1}$ can be seen as the slope of straight line through $f(x_1)$ to $f(x_2)$. (why?)

As x_2 gets closer to x_1, the straight line becomes a tangent to the graph of f (of course when the limit exists!). Hence the value of the derivative can geometrically be interpreted as the slope of the graph of the function (and not the slope of tangent line for the graph!) in the point of differentiation.

Remember, it is the slope of the graph, and not the slope of the tangent line (How can you draw a tangent line for a strait line for functions like $f(x) = mx + b$?) Although in a microscopic scale, they both coincide.

Example where derivative does not exist

Consider $f(x) = |x|$. If $x_0 > 0$, then

$$f'(x_0) = \lim_{x \to x_0} \frac{|x| - |x_0|}{x - x_0} = \lim_{x \to x_0} \frac{x - x_0}{x - x_0} = 1$$

And if $x_0 < 0$, then

$$f'(x_0) = \lim_{x \to x_0} \frac{|x| - |x_0|}{x - x_0} = \lim_{x \to x_0} \frac{-x + x_0}{x - x_0} = -1$$

For $x_0 = 0$,

$$\lim_{x \to x_0} \frac{|x| - |x_0|}{x - x_0} = \lim_{x \to 0} \frac{|x|}{x}$$

and this limit does not exist. Hence $f(x) = |x|$ is not differentiable at $x = 0$.