Advanced Calculus

Landau's Notations and Derivatives

ThinkBS: Basic Sciences in Engineering Education

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Landau's Big O and Little o notation

Let f and g be two real valued functions defined on a neighborhood of a. We say that f is big O of g around a and show it by $f(x) = O(g(x))$ as $x \to a$ if there exist positive numbers δ and M such that for all defined x with $0 < |x - a|, \delta$ we have $|f(x)| < Mg(x)$.

We say that f is little o of g around a and show it by $f(x) = o(g(x))$ as $x \to a$ if $\lim_{x \to a} \frac{f(x)}{g(x)} = 0$.

When we use the notation $o(g(x))$ without any reference to f, it means that we are talking about a function, which its limit when divided by g is equal to 0.
With this notation, the $r(h)$ in the definition of derivative is little o of h; $r(h) = o(h)$.

Examples of Landau's Notations

(1) Suppose $f(x) = \sin x$, we have $f(x) = O(1)$ as $x \to 0$ (This means sin is bounded). Also $f(x) = o(1)$ as $x \to 0$ since $\lim_{x\to 0} \frac{\sin x}{1} = 0$.

(2) Remember the Taylor expansion of e^h from session I. We can say that $e^h = 1 + h + o(h)$ as $h \to 0$. This means that e^h is equal to $1 + h$ plus a function that when divided by h its limit is 0. This can also be reformulated as:

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1$$

(3) For sin and cos, we can say that $\sin(h) = h + o(h)$ and $\cos(h) = 1 - \frac{h^2}{2} + o(h)$ as $h \to 0$. (Why?)

(1) Suppose $f(x) = e^x$, we have:

$$e^{(x+h)} = e^x.e^h = e^x.(1 + h + o(h)) = e^x + e^x.h + e^x.o(h)$$

as $h \to 0$. Take $r(h) = e^x.o(h)$, then

$$\lim_{h\to 0} |\frac{r(h)}{h}| = \lim_{h\to 0} e^x |\frac{o(h)}{h}| = e^x.0 = 0$$

. So according to the definition:

$$f'(x) = (e^x)' = e^x$$

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(2) Suppose $f(x) = \sin x$, we have:

$$\begin{aligned} \sin(x+h) &= \sin x \cos h + \cos x \sin h \\ &= \sin x.(1 - \frac{h^2}{2} + o(h)) + \cos x.(h + o(h)) \\ &= \sin x + (\cos x).h - \frac{h^2}{2}\sin x + o(h)\sin x + o(h)\cos x \end{aligned}$$

as $h \to 0$. Take $r(h) = -\frac{h^2}{2}\sin x + o(h)\sin x + o(h)\cos x$, then $\lim_{h\to 0} |\frac{r(h)}{h}| = 0$. Thus according to the definition:

$$f'(x) = (\sin x)' = \cos x$$

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(3) Suppose $f(x) = \cos x$, we have:

$$\begin{aligned}\cos(x+h) &= \cos x \cos h - \sin x \sin h \\ &= \cos x.(1 - \frac{h^2}{2} + o(h)) - \sin x.(h + o(h)) \\ &= \cos x + (-\sin x).h - \frac{h^2}{2}\cos x + o(h)\cos x - o(h)\sin x\end{aligned}$$

as $h \to 0$. Take $r(h) = -\frac{h^2}{2}\cos x + o(h)\cos x - o(h)\sin x$, then $\lim_{h\to 0}|\frac{r(h)}{h}| = 0$. Hence according to the definition:

$$f'(x) = (\cos x)' = -\sin x$$

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