Advanced Calculus

Introduction to Differential Calculus

ThinkBS: Basic Sciences in Engineering Education

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$f : \mathbb{R}^n \to \mathbb{R}^m$ will be called differentiable at point $\mathbf{x} \in \mathbb{R}^n$ if we can approximate it by a linear function around $\mathbf{x}$, or in a loose sense it is differentiable around $\mathbf{x}$ whenever it looks linear when we zoom enough.

To give a precise definition:

Definition: Suppose E is an open set in $\mathbb{R}^n$, $f : E \to \mathbb{R}^m$, and $\mathbf{x} \in E$. If there exists a linear transformation $L : \mathbb{R}^n \to \mathbb{R}^m$ (sometimes shown by df_x) such that

$$\lim_{h \to 0} \frac{|f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) - L\mathbf{h}|}{|\mathbf{h}|} = 0$$

then we say that f is differentiable at $\mathbf{x}$, and we write $f'(\mathbf{x}) = L$.

Notes on Differentiability

First note that the norm in the numerator of the definition is the norm on $\mathbb{R}^m$ and the norm in denominator is the norm on $\mathbb{R}^n$.

Next that one can show that, if such a linear transformation exists, it is unique.

How we can give meaning to this linear transformation? For a point $\mathbf{x} \in E$ we have a linear transformation defined on whole $\mathbb{R}^n$ which takes $\mathbf{h}$ as input. What kind of linear approximation is this?

To answer this, keep in mind that around $\mathbf{x} \in E$, we approximate the function f in a "direction" specified by $\mathbf{h} \in \mathbb{R}^n$, and since we can choose any direction, hence our approximation depends on the $\mathbf{h}$ at point $\mathbf{x}$.

Notes on Differentiability

So, one needs to keep in mind that when talking about differential of a function $f : E \to \mathbb{R}^m$, at a point $\mathbf{x} \in E$, we also need to talk about the "direction" given by $\mathbf{h}$.

loosely speaking, $f'(\mathbf{x}) = L$ as a linear transformation, says that in point $\mathbf{x}$ give me a direction, and I will tell you that in this point $\mathbf{x}$ and the direction of $\mathbf{h}$ how the function f is approximated.

We can also reformulate the definition as

$$f(\mathbf{x} + \mathbf{h}) = f(\mathbf{x}) + f'(\mathbf{x})\mathbf{h} + r(\mathbf{h})$$

where $\lim_{h\to 0} \frac{|r(\mathbf{h})|}{|\mathbf{h}|} = 0$.

This justifies our use of "linear approximation" around $\mathbf{x}$. (Why?)

Differentiability: Back to $\mathbb{R}$

Assume $f : E \subseteq \mathbb{R} \to \mathbb{R}$, is a real function defined on a open subset E of $\mathbb{R}$. We want to see how the definition of derivatives can be interpreted in this case.

First of all we need to see how the linear transformations $L : \mathbb{R} \to \mathbb{R}$ are. It is easy to show that every linear map $L : \mathbb{R} \to \mathbb{R}$, is in the form $L(x) = \alpha.x$ for which $\alpha \in \mathbb{R}$ is a constant. (Why?)

Let's abuse the notation and refer to α as L! Then our Linear transformation L is just a multiplication: $L(x) = Lx$. This is consistent with our notation when we used matrices for linear transformations. Here L is a 1×1 matrix and matrix multiplication is just multiplication between real numbers.

Differentiability: Back to $\mathbb{R}$

So if we also identify 1×1 matrices with real numbers, we can easily say that linear transformations $L : \mathbb{R} \to \mathbb{R}$ are just real numbers and they act on elements of $\mathbb{R}$ exactly as multiplication of real numbers.

You see where we are going? We will say that the derivative of $f : E \subseteq \mathbb{R} \to \mathbb{R}$ if exists is a **real number** such that ...

But you must never forget that this real number is in fact a linear transformation!

Differentiability: Back to $\mathbb{R}$

Now that we know about linear transformations, to be able to talk about the derivative of $f : E \subseteq \mathbb{R} \to \mathbb{R}$, we also need to know about the direction or h. But in $\mathbb{R}$ we only have one direction: Right!

This is one of the reasons why we abused the notation for linear transformations. Because in each point of differentiation we only have one choice for direction and no more, so it is better to get rid of this unnecessary technicality.

So, for the derivative of $f : E \subseteq \mathbb{R} \to \mathbb{R}$ we only need the function itself and the point $x \in E$ (the point which differentiation takes place), and it is going to be a "real number" such that:

Differentiability: Back to $\mathbb{R}$

Definition: Suppose E is an open set in $\mathbb{R}$, $f : E \subseteq \mathbb{R} \to \mathbb{R}$, and $x \in E$. We say that f is differentiable at point x, if there exists a real number α such that

$$f(x+h) = f(x) + \alpha h + r(h)$$

where $\lim_{h \to 0} |\frac{r(h)}{h}| = 0$.

We show this real number α as $f'(x)$ (to also note that it depends on x).

Examples

(1) Suppose $f(x) = x^2$, we have:

$$(x+h)^2 = x^2 + 2x.h + h^2$$

Consider $r(h) = h^2$, then clearly $\lim_{h\to 0} |\frac{r(h)}{h}| = 0$. Thus according to definition $f'(x) = 2x$.

(2) Suppose $f(x) = x^3$, we have:

$$(x+h)^3 = x^3 + 3x^2.h + 3x.h^2 + h^3$$

Consider $r(h) = 3x.h^2 + h^3$, then clearly $\lim_{h\to 0} |\frac{r(h)}{h}| = 0$. Thus according to definition $f'(x) = 3x^2$.

Examples

(3) Suppose $f(x) = x^n$, we have:

$$(x+h)^n = x^n + nx^{n-1}.h + r(h)$$

Here $r(h)$ contains all the terms in binomial expansion with h's of exponent more than 2. Clearly $\lim_{h \to 0} |\frac{r(h)}{h}| = 0$. Thus according to definition $f'(x) = nx^{n-1}$.