Advanced Calculus

Linear Algebra Basics

ThinkBS: Basic Sciences in Engineering Education

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A Reminder from Linear Algebra

As we know, $\mathbb{R}^n$ is a vector spaces over the field $\mathbb{R}$; this means that we can define an addition operation $+$ between the elements of $\mathbb{R}^n$ (this is the component-wise addition of vectors) and a scalar product . between the vectors in $\mathbb{R}^n$ and 'scalars' from $\mathbb{R}$.
Also for each element/vector in $\mathbb{R}^n$ we can assign *a real number* as its length called the norm:

First "inner product" (or scalar product) of $\mathbf{x}$ and $\mathbf{y} \in \mathbb{R}^n$ is defined by

$$\mathbf{x}.\mathbf{y} = \sum_{i=1}^{n} x_i y_i$$

Now norm of $\mathbf{x} \in \mathbb{R}^n$, *as a real number*, is defined as:

$$|\mathbf{x}| = (\mathbf{x}.\mathbf{x})^{\frac{1}{2}} = (\sum_{i=1}^{n} x_i^2)^{\frac{1}{2}}$$

Properties of Norm

If $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$ then:

- $|\mathbf{x}| \geq 0$
- $|\mathbf{x}| = 0$ iff $\mathbf{x} = 0$
- $|\alpha \mathbf{x}| = |\alpha||\mathbf{x}|$
- $|\mathbf{x}.\mathbf{y}| \leq |\mathbf{x}||\mathbf{y}|$
- $|\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$
- $|\mathbf{x} - \mathbf{z}| \leq |\mathbf{x} - \mathbf{y}| + |\mathbf{y} - \mathbf{z}|$

One should also note that, considering the properties of a norm given above, if we define $d(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|$, then we recover the Euclidean metric on $\mathbb{R}^n$.

A Reminder from Linear Algebra

Now consider two vector spaces $\mathbb{R}^n$ and $\mathbb{R}^m$. We say a function $L : \mathbb{R}^n \to \mathbb{R}^m$ is linear, if it behaves well with the addition and scalar products; in other words if for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $c \in \mathbb{R}$:

$$L(\mathbf{x} + \mathbf{y}) = L(\mathbf{x}) + L(\mathbf{y}) \qquad L(c\mathbf{x}) = cL(\mathbf{x})$$

We can also define the notion of norm for a linear transformation $L : \mathbb{R}^n \to \mathbb{R}^m$, shown by $||L||$, as the supremum of of all numbers $|L\mathbf{x}|$, where $\mathbf{x}$ ranges over all vectors in $\mathbb{R}^n$ with $|\mathbf{x}| < 1$.

A Reminder from Linear Algebra

Every linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ can be represented as a matrix multiplication. For instance $L : \mathbb{R}^3 \to \mathbb{R}^2$ with $L(x, y, z) = (x + y, 2x - y + z)$ can be written as

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Hence for linear maps, instead of writing $L(\mathbf{x})$, we simply write $L\mathbf{x}$. Matrices on the hand are very well studied and are 'simple' objects compared to other general maps $f : \mathbb{R}^n \to \mathbb{R}^m$. Strangely, for such general maps, if we zoom enough, we see that locally they behave like linear function! For instance consider the graph of $y = x^2$: as we zoom in we see that it gets very similar to a line locally!