Advanced Calculus

Rearrangement of Series

ThinkBS: Basic Sciences in Engineering Education

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Rearrangement of Series

Let $\{k_n\}_{n=1,2,3,\ldots}$ be a sequence in which every positive integer appears once and only once, in other words $\{k_n\}$ is a one-to-one and onto function from $\mathbb{N}$ to $\mathbb{N}$.

Define $a'_n = a_{k_n}$ for $n = 1, 2, 3, \ldots$. Then $\sum a'_n$ is called a rearrangement of $\sum a_n$.

If $\{s_n\}$ and $\{s'_n\}$ are the sequences of partial sums of $\sum a_n$ and $\sum a'_n$, it can be seen that, in general these two sequences consist of entirely different numbers, and may even not converge to a same limit.

Theorem: For absolutely convergent series, any rearrangement converges to the same limit.

What about conditionally convergent series?

Riemann's Theorem on Rearrangement of Series

Assume that $\sum a_n$ is a conditionally convergent real series. Take $\gamma \in \mathbb{R} \cup \{-\infty, +\infty\}$. Then there exists a rearrangement $\sum a'_n$ of $\sum a_n$ such that $\sum a'_n = \gamma$.

What a wonderful theorem!

For a proof, see 'Chap. 3: Rearrangements'.