Advanced Calculus

Power Series

ThinkBS: Basic Sciences in Engineering Education

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Series with Variables

We want to also have an additional variable in general term of our series, let's call it x, to be able create many different **numerical series** for different values of x.

The problem here is that as x changes, we have different numerical series and must for each of them see if the limit of partial sums exists or not (or if the series is convergent or divergent).

The following theorem, shows this for a special kind of series:

Theorem: If $0 \leq x < 1$, then $\sum_{n=0}^{\infty} x^n$ is convergent and its value is equal to $\frac{1}{1-x}$, in other terms:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

If $x \geq 1$, then $\sum_{n=0}^{\infty} x^n$ is divergent.

Power Series

Given a sequence $\{c_n\}$ of complex numbers, the series $\sum_{n=0}^{\infty} c_n z^n$ is called a power series. The numbers c_n are called the coefficients of the series; z is a complex number.

Theorem: For the power series $\sum_{n=0}^{\infty} c_n z^n$, put $\alpha = \lim_{n\to\infty} \sqrt[n]{|c_n|}$, and $R = \frac{1}{\alpha}$ (R is called the radius of convergence), then $\sum c_n z^n$ converges for $|z| < R$ and diverges for $|z| > R$. Nothing can be said when $|z| = R$.

Now that we know when the $\sum c_n z^n$ exists, we can change the variable z inside the *circle of convergence*, and hence we will have a function of z which is defined by a convergent series.

Example 1: For the series $\sum n^n z^n$, we have $R = 0$. So this series defines a function on only one point on the complex plane: $z = 0$ and its value on this point is $f(z) = 0$. Not a very interesting one!

Example 2: (Geometric Series) For the series $\sum z^n$, we have $R = 1$. So this series defines a function for $|z| < 1$. For such z, the values of this function is given by $\frac{1}{1-z}$.

Power Series: examples

Example 3: (Exponential Function) For the series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$, we have $R = +\infty$. So this power series defines a function on the whole complex plane. We define exponential function $\exp(z)$ as following:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad \textit{for all } z \in \mathbb{C}.$$

Example 4: (Sine Function) For the series $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$, with $x \in \mathbb{R}$, we have $R = +\infty$. So this power series defines a function on the whole real line. We define the function $\sin(x)$ as following:

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad \textit{for all } x \in \mathbb{R}.$$

Example 5: (Cosine Function) For the series $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$, with $x \in \mathbb{R}$, we have $R = +\infty$. So this power series defines a function on the whole real line. We define the function $\cos(x)$ as following:

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad \textit{for all } x \in \mathbb{R}.$$

Riemann's Zeta Function

The series $\sum_{n=1}^{\infty} \frac{1}{n^z}$ is convergent for $|\Re(z)| > 1$. (Is this a power series?). We define

$$\zeta(z) = \sum_{n=0}^{\infty} \frac{1}{n^z}$$

Riemann has shown that we can extend the definition of this function (called analytical continuation) to the whole complex plane except $z = 1$.

This function plays a key role in the study of the distribution of prime numbers.