Advanced Calculus

Sequences and Series on $\mathbb{R}$

ThinkBS: Basic Sciences in Engineering Education

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Return to Euclidean Spaces

Up to now, we have discussed about some of the properties of Euclidean spaces in the general setup of metric and topological spaces.

We have also talked about sequences and their limits, also of functions an their continuity on such general spaces.

In what follows we will talk about some special sequences, define what series are and will use them to define functions on $\mathbb{R}$.

Here are some special sequences that we encounter more often in the context of Calculus together with their limits:

- If $p > 0$, then $\lim_{n\to\infty} \frac{1}{n^p} = 0$.
- If $p > 0$, then $\lim_{n\to\infty} \sqrt[n]{p} = 1$.
- $\lim_{n\to\infty} \sqrt[n]{n} = 1$.
- If $p > 0$ and $\alpha \in \mathbb{R}$, then $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$.
- If $|x| < 1$, then $\lim_{n\to\infty} x^n = 0$.

For a proof, see 'Chap. 3: Some Special Sequences'.

Given a sequence $\{a_n\}$ we use the notation

$$\sum_{n=p}^{q} a_n \quad (p \leq q)$$

to denote the sum $a_p + a_{p+1} + \cdots + a_q$.

With $\{a_n\}$ we associate a sequence $\{s_n\}$, where

$$s_n = \sum_{m=1}^{n} a_m.$$

The numbers $\{s_n\}$ are called the partial sums of the series.

If the sequence of partial sums $\{s_n\}$ is convergent, then we show its limit by $\sum_{n=1}^{\infty} a_n$ and call $\sum a_n$ a convergent series with general terms $\{a_n\}$. Otherwise, if the limit does not exist, we say $\sum a_n$ is divergent or diverges.

For this we also use the symbolic expression $a_1 + a_2 + a_3 + \ldots$. Remember that since there is an infinity summation, this last expression is meaningless and is solely used to refer to the limit of $\{s_n\}$, which is $\sum_{n=1}^{\infty} a_n$ if it exists.

The Cauchy Criterion for Numerical Series

For a sequence $\{a_n\}$ in $\mathbb{R}$ we know that it is convergent, iff it is a Cauchy sequence. Remember that all convergent sequences were Cauchy, and since $\mathbb{R}$ is complete, every Cauchy sequence is convergent.

Since a series $\{s_n\}$ is in fact a sequence itself, we can say that:

Theorem: $\sum a_n$ converges iff for every $\varepsilon > 0$ there is an integer N_ε such that if $m \geq n \geq N_\varepsilon$, then

$$\sum_{k=n}^{m} |a_k| < \varepsilon$$

Corrolary: If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$. (Why? What about the converse?)

Theorems on Convergence of Numerical Series

- A series of nonnegative terms converges iff its partial sums form a bounded sequence.
- If $|a_n| \leq c_n$ for $n \geq N_0$, where N_0 is some fixed integer, and if $\sum c_n$ converges, then $\sum a_n$ converges.
- If $a_n \geq d_n \geq 0$ for $n \geq N_0$, and if $\sum d_n$ diverges, then $\sum a_n$ also diverges.
- (p-series) $\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p < 1$.
- The series $\sum_{n=0}^{\infty} \frac{1}{n!}$ is convergent. We show the limit of this series by e and call it Euler's number. $e = 2.71828...$.

For further information, see 'Chap. 3: The Root and Ratio Tests'.