Advanced Calculus

Connectedness

ThinkBS: Basic Sciences in Engineering Education

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Connectedness

Two subsets A and B of a metric space (X, d) are said to be **separated** if

$$\bar{A} \cap B = A \cap \bar{B} = \emptyset$$

In other words, if no point of A lies in the closure of B and no point of B lies in the closure of A.

One should remember that two sets A and B are said to be disjoint if $A \cap B = \emptyset$. Two sets can be disjoint but not separated.

Example: In the Euclidean space $\mathbb{R}$, two sets $A = [0, 1] \cap \mathbb{Q}$ and $B = [0, 1] \cap \mathbb{Q}^c$ are disjoint but not separate.

A set $E \subseteq X$ is said to be **connected** if E is not a union of two nonempty separated sets.

Clopen Sets and Connectedness

We say a set A in a metric space is clopen if it is both open and close.

In $X = (0,1) \cup (1,2)$ with metric inherited from $\mathbb{R}$, $A = (0,1)$ is a clopen set. (why?)

A metric space (X, d) is connected iff there are no proper clopen subsets in the space.

Theorem: A subset E of the real line $\mathbb{R}$ is connected iff it has the following property: If $x, y \in E$, and $x < z < y$, then $z \in E$.

This theorem states that the connected subsets of real line are the intervals.

Theorem: Assume that $f : (X, d_X) \to (Y, d_Y)$ is a continuous function. Then image of any connected set in X under f is a connected set in Y.

Totally Disconnected Sets

In a metric space (X, d) any singleton $\{x\} \subseteq X$ is clearly connected. We call these sets as trivial connected subsets of metric space (X, d).

A set A in a metric space (X, d) is called totally disconnected if it has no non-trivial connected subsets.

In $\mathbb{R}$, the sets $\mathbb{Q}$, $\mathbb{R} \setminus \mathbb{Q}$ and the Cantor set, are all examples of **totally disconnected** sets.

For further information about Cantor Set, see 'Chap. 2: Perfect Sets'.